ITC8190
Mathematics for Computer Science
Binary Relations on a Set

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A **binary relation** R on a set A is the subset

$$R \subseteq A \times A : xRy \Longleftrightarrow (x, y) \in R \ .$$

The relation $<$ on a set $A = \{1, 2, 3\}$ is the subset $\{(1, 2), (1, 3), (2, 3)\}$.

Relation R on a set A is **reflexive** if every element x in A is related to itself. It means that

$$\forall x \in A : xRx \enspace .$$

Example: the relation $\leqslant$ on $\mathbb{Z}$ is reflexive, but the relation $<$ is not.

R is called **anti–reflexive** if every element x in A is not related to itself.

$$\forall x \in A : \neg(xRx) \enspace .$$

Relation $<$ on $\mathbb{Z}$ is anti–reflexive.

Relation R on a set A is called **symmetric** if for any pair of elements x, y in A, it holds that if x is related to y, then y is related to x.

$$\forall x, y \in A : xRy \implies yRx \ .$$

Example: the relation $=$ on $\mathbb{R}$ is symmetric, since for all $a, b \in \mathbb{R}$ it holds that $a = b$ implies $b = a$.

Relation R on a set A is **anti–symmetric** if for any pair of elements x, y in A it holds that if x is related to y, and y is related to x, then x and y are the same element (written as $x = y$).

$$\forall x, y \in A : xRy \wedge yRx \implies x = y \ .$$

Example: relation $\leqslant$ is anti–symmetric, since

$$x \leqslant y \wedge y \leqslant x \implies x = y \ .$$

Relation R on a set A is **asymmetric** if it holds that if x is related to y, then y is unrelated to x.

$$\forall x, y \in A : xRy \implies \neg(yRx) \ .$$

Example: the relation $<$ on $\mathbb{R}$ is asymmetric, and the condition $x < y$ implies that $y \not< x$.

$$x < y \implies \neg(y < x) \ .$$

Relation R on a set A is **transitive** if

$$\forall x, y, z \in A : xRy \wedge yRz \implies xRz \ .$$

Example: relations $<$ and $=$ are transitive. It can be seen that

$$a < b \wedge b < c \implies a < c \ ,$$
$$a = b \wedge b = c \implies a = c \ .$$

Proposition 1

Symmetric and transitive relation is reflexive.

Proof.

By symmetry,

$$xRy \implies yRx \ .$$

By transitivity,

$$xRy \wedge yRx \implies xRx \ .$$

Therefore, symmetry and transitivity imply reflexivity. $\square$

Proposition 2

Asymmetric relation is anti-reflexive.

Proof.

By asymmetry, $xRy \implies \neg(yRx)$. Since y can be any element, let $y = x$. Then $xRx \implies \neg(xRx)$. Hence, asymmetry implies anti–reflexivity. $\square$

Proposition 3

Anti–reflexive and transitive relation is asymmetric.

Proof.

Indeed, it can be seen that $xRy \wedge yRx$ is always false. By transitivity,

$$xRy \wedge yRx \implies xRx \ ,$$

which contradicts with anti–reflexivity. So xRy and yRx cannot happen at the same time. Therefore,

$$xRy \implies \neg(yRx) \ .$$

□

Proposition 4

Anti–reflexive and transitive relation is anti-symmetric.

Proof.

By transitivity,

$$xRy \wedge yRx \implies xRx \ ,$$

which contradicts with the anti–reflexivity property. And so, the implication

$$xRy \wedge yRx \implies x = y$$

is true. □

Corollary 1

If the relation is anti–reflexive and transitive, then anti–symmetry is the same as asymmetry.

Proposition 5

Anti–reflexive relation is anti–symmetric iff it is asymmetric.

Proof.

First, we show that if anti–reflexive relation is asymmetric, then it is anti–symmetric. We need to show that $xRy \wedge yRx \implies x = y$. By assymetry, $xRy \implies \neg(yRx)$, the expression $xRy \wedge yRx$ is always false, and hence the implication $xRy \wedge yRx \implies x = y$ is true.

Secondly, we show that if anti–reflexive relation is anti–symmetric, then it is asymmetric. We need to show that $xRy \implies \neg(yRx)$. Let xRy. If yRx is true, then by anti–symmetry $xRy \wedge yRx$ would imply $x = y$. If yRx is true and $y = x$, then xRx is true. A contradiction with anti–reflexivity. And so, if xRy is true, yRx must be false. Hence $xRy \implies \neg(yRx)$. □

Relation R on a set A is **connex** if any pair of elements in A is comparable under R.

$$\forall x, y \in A : xRy \veebar yRx \ .$$

R is called **trichotomous** if any pair of elements in A is either comparable under R or is the same element.

$$\forall x, y \in A : xRy \veebar yRx \veebar x = y \ .$$

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THANK YOU
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ANY QUESTIONS?