Bachmann–Landau asymptotic notation for comparing functions

Big $\mathcal{O}$ notation

Function $f(x)$ grows no faster than function $g(x)$. Formally, function $f(x)$ is bounded by function $g(x)$ from above (written as $f(x) = \mathcal{O}(g(x))$) if

$$\exists k > 0 \ \exists x_0 \ \forall x > x_0 f(x) \leqslant k \cdot g(x) \ ,$$

or similarly

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty \ .$$

I.e., let $f(x) = 4x^2 - 2x + 2$. We can say that $f(x) = \mathcal{O}(x^2)$, since

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} = \limsup_{x \to \infty} \frac{4x^2 - 2x + 2}{x^2} = \limsup_{x \to \infty} 4 - \frac{2}{x} + \frac{2}{x^2} = 4 < \infty \ .$$

Big Ω notation

Function $f(x)$ grows not slower than function $g(x)$. Formally, function $f(x)$ is bounded from below by function $g(x)$ (written as $f(x) = \Omega(g(x))$) if

$$\exists k > 0 \ \exists x_0 \ \forall x > x_0 f(x) \geqslant k \cdot g(x) \ ,$$

or similarly

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0 \ .$$

I.e., let $f(x) = 4x^2 - 2x + 2$. We can say that $f(x) = \Omega(x^2)$, since

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} = \limsup_{x \to \infty} \frac{4x^2 - 2x + 2}{x^2} = \limsup_{x \to \infty} 4 - \frac{2}{x} + \frac{2}{x^2} = 4 > 0 \ .$$

Big Θ notation

Function $f(x)$ grows not faster and not slower than function $g(x)$. Formally, function $f(x)$ is bounded from both sides by function $g(x)$ (written as $f(x) = \Theta(g(x))$) if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Theta(g(x))$.

I.e., let $f(x) = 2x^3 - 7x + 1$. We can say that $f(x) = \Omega(x^3)$, since

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} = \limsup_{x \to \infty} \frac{2x^3 - 7x + 1}{x^3} = \limsup_{x \to \infty} 2 - \frac{7}{x} + \frac{1}{x^3} = 2 \ ,$$

and $0 < 2 < \infty$.

Small o notation

Function $g(x)$ grows faster than function $f(x)$. Formally, function $f(x)$ is dominated by function $g(x)$ asymptotically (written as $f(x) = o(g(x))$) if

$$\forall k > 0 \ \exists x_0 \ \forall x > x_0 f(x) < k \cdot g(x) \ \ ,$$

or similarly

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} = 0 \ \ .$$

I.e., let $f(x) = 4x^2 - 2x + 2$. We can say that $f(x) = o(x^3)$, since

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} = \limsup_{x \to \infty} \frac{4x^2 - 2x + 2}{x^3} = \limsup_{x \to \infty} \frac{4}{x} - \frac{2}{x^2} + \frac{2}{x^3} = 0 \ \ .$$

Small ω notation

Function $f(x)$ grows faster than function $g(x)$. Formally, function $f(x)$ asymptotically dominates function $g(x)$ (written as $f(x) = \omega(g(x))$) if

$$\forall k > 0 \ \exists x_0 \ \forall x > x_0 f(x) > k \cdot g(x) \ \ ,$$

or similarly

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} = \infty \ \ .$$

I.e., let $f(x) = 2x^3 - 7x + 1$. We can say that $f(x) = \omega(x^2)$, since

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} = \limsup_{x \to \infty} \frac{2x^3 - 7x + 1}{x^2} = \limsup_{x \to \infty} 2x - \frac{7}{x} + \frac{1}{x^2} = \infty \ \ .$$