Exercise 1. Let p, q, r, s, t, u be integers, where q, s, u are non-zero. A relation R is defined by

$$\frac{p}{q} \sim \frac{r}{s} \Longleftrightarrow ps = qr \ .$$

Show that R is an equivalence relation.

Solution. Clearly, $\sim$ is reflexive, since

$$\frac{p}{q} \sim \frac{p}{q} \Longleftrightarrow pq = pq \ ,$$
$$\frac{q}{r} \sim \frac{q}{r} \Longleftrightarrow qr = qr \ .$$

$\sim$ is symmetric, since

$$\frac{p}{q} \sim \frac{r}{s} \Longleftrightarrow \frac{r}{s} \sim \frac{p}{q}$$
$$ps = qr \Longleftrightarrow rq = ps \ .$$

$\sim$ is also transitive. To show this consider

$$\frac{p}{q} \sim \frac{r}{s} \sim \frac{t}{u} \ .$$

Then $ps = qr$ and $ru = st$. Therefore,

$$\frac{ps}{q} = r = \frac{st}{u}$$
$$\frac{ps}{q} = \frac{st}{u}$$
$$psu = stq$$

Since $s \neq 0$, we have $pu = qt$. Consequently, $\frac{p}{q} \sim \frac{t}{u}$. Two pairs of integers (p, q) and (r, s) are in the same equivalence class when they reduce to the same fraction in its lowest terms.

Exercise 2. For (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$, relation R is defined by

$$(x_1, y_1) \sim (x_2, y_2) \Longleftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2 \ .$$

Show that R is an equivalence relation.

Solution. Indeed, $\sim$ is reflexive, since

$$(x_1, y_1) \sim (x_1, y_1) \Longleftrightarrow x_1^2 + y_1^2 = x_1^2 + y_1^2 \ ,$$
$$(x_2, y_2) \sim (x_2, y_2) \Longleftrightarrow x_2^2 + y_2^2 = x_2^2 + y_2^2 \ .$$

$\sim$ is symmetric, since

$$(x_1, y_1) \sim (x_2, y_2) \Longleftrightarrow (x_2, y_2) \sim (x_1, y_1) \ ,$$
$$x_1^2 + y_1^2 = x_2^2 + y_2^2 \Longleftrightarrow x_2^2 + y_2^2 = x_1^2 + y_1^2 \ .$$

$\sim$ is transitive, since

$$(x_1, y_1) \sim (x_2, y_2) \sim (x_3, y_3) \Longleftrightarrow (x_1, y_1) \sim (x_3, y_3) \ ,$$
$$x_1^2 + y_1^2 = x_2^2 + y_2^2 \text{ and } x_2^2 + y_2^2 = x_3^2 + y_3^2 \Longleftrightarrow x_1^2 x_1^2 + y_1^2 = x_3^2 + y_3^2 \quad \Longleftrightarrow x_1^2 + y_1^2 = x_3^2 + y_3^2 \ ,$$

Two pairs of real numbers are in the same equivalence class when they lie on the same circle around the origin.

Exercise 3. Determine whether or not the following relations are equivalence relations on the given set. Show which properties of an equivalence relations hold and which not.

Solution. (a) $x \sim y$ in $\mathbb{R}$ if $x \geqslant y$.

- Reflexive: $x \geqslant x$ and $y \geqslant y$.
- Anti-symmetric: $x \geqslant y \wedge y \geqslant x \implies x = y$.
- Transitive: $x \geqslant y \wedge y \geqslant z \implies x \geqslant z$.

(b) $m \sim n$ in $\mathbb{Z}$ if $mn > 0$.

- Reflexive: $mm \geqslant 0$.
- Symmetric: $mn \geqslant 0 \implies nm \geqslant 0$.
- Transitive: $mn \geqslant 0 \wedge nk \geqslant 0 \implies mk \geqslant 0$.

(c) $x \sim y$ in $\mathbb{R}$ if $|x - y| \leqslant 4$.

- Reflexive: $|x - x| \leqslant 4$.
- Symmetric: $|x - y| \leqslant 4 \implies |y - x| \leqslant 4$, since $|x - y| = |y - x|$.
- Not transitive: $|x - y| \leqslant 4 \wedge |y - z| \leqslant 4 \not\Longrightarrow |x - z| \leqslant 4$. A counter–example: $x = 8, y = 4, z = 1$. We have $|8 - 4| \leqslant 4 \wedge |4 - 1| \leqslant 4 \not\Longrightarrow |8 - 1| = 7 \leqslant 4$.

(d) $m \sim n$ in $\mathbb{Z}$ if $m \equiv n \pmod 6$.

- Reflexive: $m \equiv m \pmod 6$.
- Symmetric: $m \equiv n \pmod 6 \implies n \equiv m \pmod 6$. If $n|(m - n)$, then also $n|(n - m)$.
- Transitive: $m \equiv n \pmod 6 \wedge n \equiv k \pmod 6 \implies m \equiv k \pmod 6$. There exist $\alpha, \beta \in \mathbb{Z} : m = 6\alpha + n$ and $n = 6\beta + k$. This means that $m = 6(\alpha + \beta) + k$, hence $m \equiv k$.

Exercise 4. Define a relation $\sim$ on $\mathbb{R}^2$ by stating that

$$(a, b) \sim (c, d) \Longleftrightarrow a^2 + b^2 \leqslant c^2 + d^2 \ .$$

Show that $\sim$ is reflexive, transitive, but not symmetric.

Solution. $\sim$ is reflexive, since

$$(a, b) \sim (a, b) \Longleftrightarrow a^2 + b^2 \leqslant a^2 + b^2 \ .$$

$\sim$ is anti-symmetric, since

$$(a,b) \sim (c,d) \wedge (c,d) \sim (a,b) \implies (a,b) = (c,d) \ ,$$
$$a^2+b^2 \leqslant c^2+d^2 \wedge c^2+d^2 \leqslant a^2+b^2 \implies a^2+b^2 = c^2+d^2 \ .$$

$\sim$ is transitive, since

$$(a,b) \sim (c,d) \sim (e,f) \implies (a,b) \sim (e,f) \ ,$$
$$a^2+b^2 \leqslant c^2+d^2 \leqslant e^2+f^2 \implies a^2+b^2 \leqslant e^2+f^2 \ .$$

Exercise 5 (Projective Real Line $\mathbb{P}(\mathbb{R})$)**.** Define a relation on $\mathbb{R}^2 \setminus (0,0)$:

$$(x_1,y_1) \sim (x_2,y_2) \iff \exists \lambda \in \mathbb{R}, \lambda \neq 0 : (x_1,y_1) = (\lambda x_2, \lambda y_2) \ .$$

Show that $\sim$ defines an equivalence relation on $\mathbb{R}^2 \setminus (0,0)$.

Solution. $\sim$ is reflexive:

$$(x_1,y_1) \sim (x_1,y_1) \iff (x_1,y_1) = (1 \cdot x_1, 1 \cdot y_1) \ .$$

$\sim$ is symmetric:

$$(x_1,y_1) \sim (x_2,y_2) \iff (x_2,y_2) \sim (x_1,y_1) \ ,$$
$$(x_1,y_1) = (\lambda x_2, \lambda y_2) \iff (x_2,y_2) = (\frac{1}{\lambda}x_1, \frac{1}{\lambda}y_1) \ .$$

To show that $\sim$ is transitive, assume $(x_1,y_1) \sim (x_2,y_2)$ and $(x_2,y_2) \sim (x_3,y_3)$. We need to show that $(x_1,y_1) \sim (x_3,y_3)$.

$$(x_1,y_1) = (\lambda x_2, \lambda y_2) \ ,$$
$$(x_2,y_2) = (\gamma x_3, \gamma y_3) \ ,$$
$$(x_1,y_1) = (\lambda x_2, \lambda y_2) = (\lambda\gamma x_3, \lambda\gamma y_3) \ .$$

Hence, $(x_1,y_1) \sim (x_3,y_3)$ and so $\sim$ is reflexive, symmetric and transitive and hence is an equivalence relation on $\mathbb{R} \setminus (0,0)$.

Exercise 6. Let $\mathbb{Z}*$ be the set of all non-zero integers, and let R be a relation on $\mathbb{Z} \times \mathbb{Z}*$ given by

$$\forall x,y \in \mathbb{Z}, \forall x',y' \in \mathbb{Z}* : (x,y)R(x',y') \iff xy' = x'y \ .$$

Show that R is an equivalence relation.

Solution. R is reflexive, since $(x,y) = (x,y) \implies xy = xy$. R is symmetric since $(x,y) = (x'y')$ if $xy' = x'y$, as well as $(x'y') = (x,y)$ if $x'y = xy'$. Finally, to show that R is transitive, consider the case when $(x,y) = (x',y') = (x'',y'')$. This means that

$$(x,y) = (x',y') \implies xy' = x'y \ ,$$
$$(x',y') = (x'',y'') \implies x'y'' = x''y' \ .$$

We need to show that $(x,y) = (x'',y'') \implies xy'' = x''y$. From equality $xy' = x'y$ we extract $y' = \frac{x'y}{x}$ and substitute y' with it in equality $x'y'' = x''y'$ to obtain

$$x'y'' = x''y' \implies x'y'' = \frac{x''x'y}{x} \implies xx'y'' = x''x'y \implies xy'' = x''y \ .$$