ITC8190
Mathematics for Computer Science
Mappings and their properties

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A **binary relation** R between sets A and B is the subset

$$R \subseteq A \times B : \forall x \in A, \forall y \in B : xRy \iff (x, y) \in R \ .$$

A binary relation is a **mapping** (or a **function**) $f \colon A \to B$ if it is functional (right-unique) and left-total.

In other words, $R \subseteq A \times B$ maps every element $a \in A$ to a *unique* element $b \in B$.

An **injection** is an injective mapping – a binary relation that is left-unique, right-unique, and left-total

A **surjection** (or **onto mapping**) is a surjective mapping – a binary relation that is right-unique, left-total, and right-total.

A mapping is a **bijection** (or **one-to-one correspondence**) is a mapping which is injective and surjective. In other words, left-unique, right-unique, left-total, and right-total.

A **linear mapping** or **linear transformation** is a map $\mathbb{R}^n \to \mathbb{R}^m$ given by a matrix.

For example, given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,$$

we can define a map $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\forall (x, y) \in \mathbb{R}^2 : \ T_A(x, y) = (ax + by, cx + dy) \ .$$

This is actually matrix multiplication, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \ .$$

For any set S, a bijective mapping $\pi : S \to S$ is called a **permutation**.

Suppose $S = \{1, 2, 3\}$. Define a map $\pi : S \to S$ by

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} .$$

It is easy to verify that this map is bijective, hence this map is a permutation of S.

Let S be a set. The **identity map** id_S is such that

$$\forall s \in S : s \mapsto s \ .$$

In example, for $S = \{1, 2, 3\}$, the identity map id_S is

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \ .$$

A **composition of mappings** $f : A \to B$ and $g : B \to C$ is a new mapping $h : A \to C$ defined by

$$(g \circ f)(x) = g(f(x)) \quad .$$

Note that $g(f(x)) = (g \circ f)(x) \neq (f \circ g)(x) = f(g(x))$.

Consider the following sets

$$A = \{1, 2, 3\} \qquad B = \{a, b, c\} \qquad C = \{x, y, z\} \ .$$

Consider mappings

$$f \colon A \to B \text{ defined by } \{1 \mapsto b, 2 \mapsto c, 3 \mapsto a\} \ ,$$
$$g : B \to C \text{ defined by } \{a \mapsto z, b \mapsto z, c \mapsto x\} \ .$$

The composition $g \circ f \colon A \to C$ is defined by $\{1 \mapsto z, 2 \mapsto x, 3 \mapsto z\}$.

What can you say about the composition $f \circ g$?

Theorem 1

The composition of mappings in associative. That is, for $f : A \to B, g : B \to C$, *and* $h : C \to D$:

$$(h \circ g) \circ f = h \circ (g \circ f) \quad .$$

Proof.

Let $a \in A$. Then

$$\begin{aligned}(h \circ (g \circ f))(a) &= h((g \circ f)(a)) = h(g(f(a))) \\ &= (h \circ g)(f(a)) = ((h \circ g) \circ f)(a) \quad .\end{aligned}$$

□

Let $f\colon A \to B$ be a mapping. The **inverse mapping** $f^{-1} : B \to A$ is a mapping such that

$$f \circ f^{-1} = id_B \ ,$$
$$f^{-1} \circ f = id_A \ .$$

A mapping $f\colon A \to B$ is **invertible** (has a corresponding inverse mapping) iff f is bijective.

The mapping $f\colon \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln(x)$ has an inverse $f^{-1}(x) = e^x$.

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(e^x) = \ln e^x = x \ ,$$
$$(f^{-1} \circ f)(x) = f^{-1}(\ln x) = e^{\ln x} = x \ .$$

To show that a mapping is invertible iff it is bijective, we need the following lemmas.

Lemma 1

An invertible mapping is injective.

Proof.

Suppose that $f: A \to B$ is invertible with inverse $f^{-1} : B \to A$. Then

$$\begin{aligned} \forall a, b \in A : f(a) = f(b) &\implies f^{-1}(f(a)) = f^{-1}(f(b)) \\ &\implies (f^{-1} \circ f)(a) = (f^{-1} \circ f)(b) \\ &\implies id_A(a) = id_A(b) \\ &\implies a = b \ . \end{aligned}$$

Consequently, f is injective. □

Lemma 2

An invertible mapping is surjective.

Proof.

Suppose that $f\colon A \to B$ is invertible with inverse $f^{-1} : B \to A$. Suppose that $b \in B$. To show that f is surjective, for every $b \in B$ we need to find $a \in A$ such that $f(a) = b$. Indeed, such an a exists:

$$\forall b \in B\colon\ \exists a = f^{-1}(b) \in A : f(f^{-1}(b)) = (f \circ f^{-1})(b) = b\ .$$

Consequently, f is surjective. □

Theorem 2

A mapping $f\colon A \to B$ is invertible iff it is bijective.

Proof.

By Lemmas 1 and 2, an invertible mapping is bijective.

To complete the proof, we will show that any bijective mapping is invertible.

Assume that $f\colon A \to B$ is bijective, and let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Because f is injective, such a must be unique. Define $f^{-1} : B \to A$ by letting $f^{-1}(b) = a$.

We have now constructed the inverse of f, hence f is invertible. □

Theorem 3

If $f : A \to B$ and $g : B \to C$ are both injective, then the mapping $g \circ f$ is injective.

Proof.

Indeed, since both f and g are injective, then for all $a, b \in A$ it holds that

$$\begin{aligned}(g \circ f)(a) = (g \circ f)(b) &\implies g(f(a)) = g(f(b)) \\ &\implies f(a) = f(b) \implies a = b \ .\end{aligned}$$

Therefore, $g \circ f$ is an injective mapping. □

Theorem 4

If $f : A \to B$ and $g : B \to C$ are both surjective, then the mapping $g \circ f$ is surjective.

Proof.

We need to show that the mapping $g \circ f : A \to C$ is surjective, or, in other words, we need to show that for every $c \in C$ there exists $a \in A$ such that $(g \circ f)(a) = c$.

Since g is surjective, there exists $b \in f(A)$ such that $g(b) = c$. In turn, surjectivity of f implies that there exists $a \in A$ such that $f(a) = b$.

Hence, for every $c \in C$ there exists $a \in A$ such that $(g \circ f)(a) = c$. □

Corollary 1

If $f : A \to B$ *and* $g : B \to C$ *are bijective, so is their composition* $g \circ f$*.*

Proof.

This is a direct consequence of Theorems 3 and 4. □

Corollary 2

The composition of permutations is a permutation.

Proof.

This is a direct consequence of Theorems 3 and 4. □

THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?