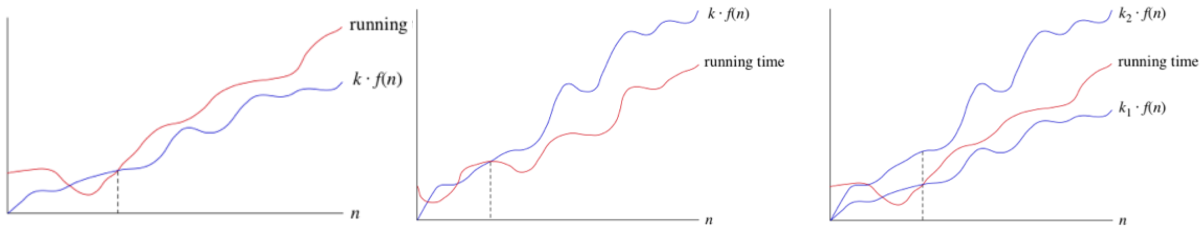
Bachmann–Landau Infinitary Asymptotic Complexity Notation



(a) Growth rate is $\Omega(f(n))$ (b) Growth rate is $O(f(n))$ (c) Growth rate is $\Theta(f(n))$

Figure 1: Asymptotic complexity illustrations [1]

Big O notation

The assertion $f(n) = O(g(n))$ means that $f(n)$ asymptotically grows at most as fast as $g(n)$. It provides an asymptotic upper bound, without specifying a lower bound. See Fig. 1b. It means that $\exists c > 0\ \exists n_0\ \forall n > n_0 : f(n) \leqslant c \cdot g(n)$, or

$$\limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty \ .$$

Ω notation

The assertion $f(n) = \Omega(g(n))$ means that $f(n)$ asymptotically grows at least as fast as $g(n)$. It provides an asymptotic lower bound without specifying an upper bound. See Fig. 1a. It means that $\exists c > 0\ \exists n_0 :\ \forall n > n_0 : f(n) \geqslant c \cdot g(n)$, or

$$\liminf_{n\to\infty} \frac{|f(n)|}{g(n)} > 0 \ .$$

Θ notation

The assertion $f(n) = \Theta(g(n))$ means that $f(n)$ is asymptotically bounded from above and from below by $g(n)$. See Fig. 1c. It means that $\exists c_1, c_2 > 0\ \exists\ n_0 \forall n > n_0 : c_1 \cdot g(n) \leqslant f(n) \leqslant c_2 \cdot g(n)$. In other words, $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, or

$$0 < \liminf_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty \ .$$

Little o notation

The assertion $f(n) = o(g(n))$ means that $g(n)$ asymptotically grows much faster than $f(n)$. It means that $\forall c > 0\ \exists n_0\ \forall n > n_0 : |f(n)| < c \cdot g(n)$, or

$$\lim_{n\to\infty} \frac{|f(n)|}{g(n)} = 0 \ .$$

[1] Images taken from `https://www.khanacademy.org/computing/computer-science/algorithms/asymptotic-notation/a/asymptotic-notation`

Note that $f(n) = o(g(n))$ implies $f(n) = O(g(n))$, but the converse is not true.

Little ω notation

The assertion $f(n) = \omega(g(n))$ means that $f(n)$ asymptotically grows much faster than $g(n)$. It means that $\forall c > 0\ \exists n_0\ \forall n > n_0 : |f(n)| > c \cdot |g(n)|$, or

$$\lim_{n\to\infty} \left| \frac{f(n)}{g(n)} \right| = \infty \ .$$

$\sim$ notation

The assertion $f(n) \sim g(n)$ means that $f(n)$ is asymptotically equal (grows as fast as) $g(n)$. It means that

$$\exists c > 0\ \exists n_0\ \forall n > n_0 : \left| \frac{f(n)}{g(n)} - 1 \right| < c \ , \qquad \text{or} \qquad \lim_{n\to\infty} \left| \frac{f(n)}{g(n)} \right| = 1 \ .$$