Exercises

Exercise 1. Consider a set $A = \{2, 4, 5, 11\}$ ordered by $\leqslant$. Is $\leqslant$ a total order or a partial order on A? What are the minimal/least/maximal/greatest elements? Let $A \subset \mathbb{N}$. What are the bounds?

Solution. $\leqslant$ is a total order on A, since every two integers are comparable under $\leqslant$. Element 2 is the minimal and least element. Element 11 is the maximal and greatest element. The lower bounds are $\{0, 1, 2\}$, and $\inf A = 2$. Upper bounds are $\{11, 12, 13, \ldots\}$ and $\sup A = 11$.

Exercise 2. Consider a set $A = \{2, 4, 5, 11\}$ ordered by divisibility $|$. Is $|$ a total order or a partial order on A? What are the minimal/least/maximal/greatest elements? Let $A \subset \mathbb{N}$. What are the bounds?

Solution. The divisibility relation is a partial order on A. I.e., 2 does not divide 5 and 5 does not divide 2. There are 3 minimal elements: $2, 5, 11$, and no least element. A least element in this context would mean an element of a which divides all other elements of A. Since A contains uncomparable elements, it the existence of such an element is impossible, and hence there is no least element. There are also 3 maximal elements: $4, 5, 11$, no greatest element (for the same reasons as there is no least element). The greatest element would be an element which is divided by every other element in A. Since 1 divides all the elements, the lower bound is 1, and since it is the only lower bound, it is an infimum. The upper bounds are $\{0, lcm(2, 4, 5, 11), 2\, lcm(2, 4, 5, 11), \ldots\}$ The supremum is $lcm(2, 4, 5, 11)$.

Exercise 3. Draw the Hasse diagram of the powerset of $\{a, b, c\}$ ordered by inclusion $\subseteq$. Is the relation $\subseteq$ a total order or a partial order on $\{a, b, c\}$?

Solution. The powerset of $\{a, b, c\}$ is the set of all sets

$$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \quad .$$

Every element is trivially a subset of itself. The other elements, ordered by inclusion, are:

$$\begin{array}{lll}
\emptyset \subseteq \{a\} \ , & \emptyset \subseteq \{b\} \ , & \emptyset \subseteq \{c\} \ , \\
\{a\} \subseteq \{a, b\} \ , & \{a\} \subseteq \{a, c\} \ , & \{b\} \subseteq \{a, b\} \ , \\
\{b\} \subseteq \{b, c\} \ , & \{a, b\} \subseteq \{a, b, c\} \ , & \{b, c\} \subseteq \{a, b, c\} \ .
\end{array}$$

Relation $\subseteq$ is a partial order relation on A, since $\{a\} \not\subseteq \{b\}$ and $\{b\} \not\subseteq \{a\}$.

Exercise 4. Draw the Hasse diagram of the set $A = \{1, 2, 3, 4, 5, 6\} \subset \mathbb{N}$ ordered by divisibility $|$. Is $|$ a total order to a partial order on A?

Solution. $|$ is a partial order on A, since $2 \not| 3$ and $3 \not| 2$. The elements ordered by divisibility are

$$1|2|4 \qquad\qquad 1|5 \qquad\qquad 1|3|6 \ .$$

Exercise 5. Consider the set $\mathbb{N} \subset \mathbb{Z}$ ordered by $\leqslant$. Is there a minimal/maximal/least/greatest element? Is the set $\mathbb{N}$ bounded? What are the bounds?

Solution. 0 is the only minimal and least element of $\mathbb{N}$ (recall that $\mathbb{N}$ is a totally–ordered set). There is no maximal or greatest element. The set $\mathbb{N}$ is bounded from below, 0 is the infimum, and any negative integer is a lower bound of $\mathbb{N}$.

Exercise 6. Consider a subset $[a, b] \subset \mathbb{N}$ ordered by $\leqslant$. Is there a minimal/maximal/least/greatest element? Is the set bounded? What are the bounds?

Solution. Since $\leqslant$ is a total order on $\mathbb{N}$, its restriction to the subset $[a, b]$ is also a total order. Hence $[a, b]$ is a totally ordered subset. a is the minimal and least element. b is the maximal and greatest element. The subset is bounded from both sides. The lower bounds are $\{a, a-1, a-2, \ldots, 0\}$ with a being the infimum, and the upper bounds are $\{b, b+1, b+2, \ldots\}$ with b being the supremum.

Exercise 7. Consider the set $\mathbb{Z}$ ordered by $\leqslant$. What are the minimal/maximal/least/greatest elements? What are the bounds?

Solution. The relation $\leqslant$ is a total order on $\mathbb{Z}$. There are no minimal nor least elements. There are no maximal no greatest elements. The set is unbounded, it does not have lower nor upper bounds, infimum nor supremum.

Exercise 8. Consider the subset $\mathbb{Z}^+$ of positive integers. What are the minimal/maximal/least/greatest elements? What are the bounds?

Solution. The subset $\mathbb{Z}^+$ is bounded from below. It has the least element 1, which is also the minimal element. The lower bounds are $\{1, 0, -1, -2, \ldots\}$, with 1 being the infimum. There is no maximal nor greatest element. There are no upper bounds, there is no supremum.

Exercise 9. Consider the subset $(\sqrt{2}, 5] \subset \mathbb{Q}$. What are the minimal/maximal/least/greatest elements? What are the bounds?

Solution. The subset $(\sqrt{2}, 5] \subset \mathbb{Q}$ is bounded from above and from below. Element $5 \in \mathbb{Q}$ is the maximal and greatest element, as well as an upper bound and supremum. Other upper bounds are, in example, $6, \frac{13}{2}, 10$. There is no minimal nor least element. Lower bounds are 0 and 1, in example. There is no infimum.

Exercise 10. Consider the set $\mathbb{C}$. What are the minimal/maximal/least/greatest elements? What are the bounds?

Solution. The set of complex number $\mathbb{C}$ is not ordered. There are no minimal/maximal/least/greatest elements, no bounds, no supremum nor infimum.

Exercise 11. Show that any real number $m \in \mathbb{R}$ is an upper and lower bound for an empty set $\emptyset$.

Solution. Let m be an upper bound of $\emptyset$. Then $s \in \emptyset \implies s \leqslant m$, which is true.

Similarly, let m be the lower bound of $\emptyset$. Then $s \in \emptyset \implies m \leqslant s$, which is also true.

Everything is true about the members of the emptyset.