Exercises

Exercise 1. (Reimo Palm) How many ways to ascend a ladder with n rungs if on each step we may advance by 1 or 2 rungs?

Solution. Let A_n be the number of ways to ascend an n-rung ladder.

If we go 1 rung on the first step, we need to climb the remaining $n-1$ rungs with the following steps, and there are A_{n-1} ways to do that. If we go 2 rungs on the first step, we need to climb the remaining $n-2$ rungs with the following steps, and there are A_{n-2} ways for that. Thus, from the rule of sum we have $A_n = A_{n-1} + A_{n-2}$, just like in the Fibonacci sequence, and so we already know the solution must have the form $A_n = c_1 q_1^n + c_2 q_2^n$, where $q_1 = \frac{1+\sqrt{5}}{2}$ and $q_2 = \frac{1-\sqrt{5}}{2}$.

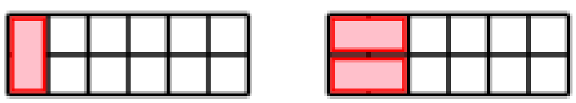
The boundary conditions are $A_1 = 1$ (we always have to ascend a 1-rung ladder in a single step) and $A_2 = 2$ (we can ascend a 2-rung ladder in one step of two rungs or two steps of one rung each). We could construct and solve the equations for c_1 and c_2 from scratch, but as a shortcut we can also notice that $A_1 = F_2$ and $A_2 = F_3$. As any second-order linear recurrence is uniquely determined by the recurrent rule and two consecutive elements, we have $A_n = F_{n+1}$.

This also gives $A_0 = 1$, which may be interpreted as there being just one way for ascending a 0-rung ladder (do nothing), though the notion of a 0-rung ladder is admittedly somewhat silly.

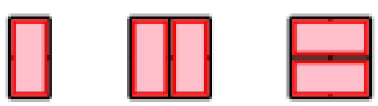
Exercise 2. (Reimo Palm) How many ways to cover a $2 \times n$ rectangle with 2×1 domino tiles so that each square of the rectangle is covered by exactly one square of a tile?

Solution. Let D_n be the number of ways a $2 \times n$ rectangle can be covered.

Consider now the top-left corner of a rectangle. It may be covered by the top half of a tile oriented vertically (on the left in the figure below). Then a $2 \times (n-1)$ rectangle remains to be covered, and there are D_{n-1} ways to do that. Alternatively, the top-left corner may be covered by the left half of a tile oriented horizontally. Then the only possibility to cover the bottom-left corner is with another tile oriented horizontally (on the right in the figure), leaving a $2 \times (n-2)$ rectangle still to be covered, with D_{n-2} ways to do that. Therefore, the recurrent rule is $D_n = D_{n-1} + D_{n-2}$.



To obtain boundary conditions, we note that there is just one way to cover a 2×1 rectangle (on the left in the figure below) and two ways to cover a 2×2 rectangle (in the middle and on the right on the figure). Therefore $D_1 = 1$ and $D_2 = 2$.



Since both the recurrent rule and the boundary conditions are the same as in the previous exercise, the two sequences must be the same, and we have $D_n = A_n = F_{n+1}$.

Exercise 3. (Reimo Palm) How many n-letter strings consisting of letters A and B, where there are never two consequtive A's?

Solution. Let S_n be the number of n-letter strings of the required form.

If the first letter of an n-letter string is B, the $(n-1)$-letter sufix may be any string of the required form. If the first letter is A, the second one has to be B (else we would have two consequtive A's) and the $(n-2)$-letter sufix may be any string of the required form. So we have $S_n = S_{n-1} + S_{n-2}$ once again.

The boundary conditions are $S_0 = 1$ (there's one empty string and it satisfies the requirement) and $S_1 = 2$ (the one-letter strings A and B both satisfy the requirement).

Therefore we have $S_n = F_{n+2}$.

Exercise 4. (Reimo Palm) How many n-letter strings consisting of letters A, B, C, and D, where there's an odd number of A's?

Solution. Let S_n be the number of n-letter strings of the required form.

If the first letter of an n-letter string is A, then the remaining $n-1$ letters must contain an even number of A's. We can note that when we take all strings and remove those with odd number of A's, exactly the ones with even number of A's remain. There are 4^{n-1} tuples of length $n-1$ in the 4-element letter set we have, so the number of $(n-1)$-letter strings containing an even number of A's must be $4^{n-1} - S_{n-1}$ and each of them can be prepended with an A to obtain an n-letter string containing an odd number od A's.

If the first letter is B, C, or D, the remaining $n-1$ letters must contain an odd number of A's. By the rule of product, we have $3S_{n-1}$ ways to construct the n-letter string starting with one of those three letters. Using the rule of sum to combine this with the result for strings starting with an A, we obtain the recurrence $S_n = 4^{n-1} - S_{n-1} + 3S_{n-1} = 2S_{n-1} + 4^{n-1}$.

For boundary condition we can use either $S_0 = 0$ (the one and only empty string has an even number of A's, so the number of 0-length strings with odd number of A's is 0) or $S_1 = 1$ (the only 1-letter string with odd number of A's is A itself).

Picking the first boundary condition, we obtain the non-homogeneous first-order linear recurrence

$$S_0 = 0,$$
$$S_{n+1} = 2S_n + 4^n.$$

Following the 3-step method for solving non-homogeneus recurrences:

- The corresponding homogeneus recurrence is $S'_{n+1} = 2S'_n$, whose characteristic equation $q - 2 = 0$ yields $q = 2$ and therefore the general solution of the homogeneus recurrence is

$$S'_n = cq^n = c2^n.$$

- To obtain a particular solution for the non-homogeneus recurrence, we look among expressions generalizing the non-homogeneus member of the recurrent rule: $S''_n = \alpha 4^n$. Substituting into the recurrent rule, we obtain $\alpha 4^{n+1} = 2\alpha 4^n + 4^n$. Dividing both sides of the equation by 4^n, we get $4\alpha = 2\alpha + 1$, which gives $\alpha = \frac{1}{2}$ and therefore a particular solution of the non-homogeneus recurrence is

$$S''_n = \frac{1}{2} 4^n.$$

- Combining the previous two results, we get that the general solution must have the form $S_n = S'_n + S''_n = c2^n + \frac{1}{2}4^n$. Looking at the boundary condition, we have $S_0 = c2^0 + \frac{1}{2}4^0 = c + \frac{1}{2} = 0$, from which we obtain $c = -\frac{1}{2}$, which in turn gives us the final result

$$S_n = -\frac{1}{2}2^n + \frac{1}{2}4^n = \frac{1}{2}(4^n - 2^n).$$

Exercise 5. (Edouard Lucas) Tower of Hanoi is a puzzle consisting of three pegs and n distinct-sized disks initially stacked in the order of sizes on the leftmost peg. The goal is to move all disks to the rightmost peg obeying the additional constraints that we may only move one disk at a time and may never place a larger disk on top of a smaller one. The middle peg can be used as a temporary holding place. How many steps are needed to solve the puzzle for n disks?

Solution. In this problem, we need to first design a solution strategy, then convince ourselves that it is indeed optimal, and only then can count the steps.

For $n = 1$, the optimal solution is quite obvious: we just take the single disk and move it from the leftmost peg to the rightmost one in 1 step. So, the number of steps $H_1 = 1$ in this case.

For $n > 1$, we can use the following process: first we move the $n-1$ smaller disks to the middle peg in H_{n-1} steps, then the largest disk from the leftmost peg to the rightmost one in 1 step, and then the $n-1$ smaller disks from the middle peg to the rightmost one in another H_{n-1} steps.

To see that this is the optimal strategy, let's consider the largest disk. In order to be able to move it at all, we need to get the $n-1$ smaller disks away from the leftmost peg, so we must spend at least H_{n-1} steps before we can move the largest disk. Symmetrically, after we have placed the largest disk to the rightmost peg, we must spend at least H_{n-1} steps to get the smaller disks on top of it. As our strategy achieves these lower bounds, it is indeed optimal, and so we have obtained the following recurrence for the number of steps required:

$$H_1 = 1,$$
$$H_{n+1} = 2H_n + 1.$$

Following the 3-step method again:

- The corresponding homogeneus recurrence is $H'_{n+1} = 2H'_n$, whose characteristic equation $q - 2 = 0$ yields $q = 2$ and therefore the general solution of the homogeneus recurrence is

$$H'_n = c2^n.$$

- Generalizing the non-homogeneus member to $H''_n = \alpha$ and substituting into the recurrent rule, we obtain $\alpha = 2\alpha + 1$, from which we have $\alpha = -1$ and therefore

$$H''_n = -1.$$

- Then the general solution must have the form $H_n = c2^n - 1$ and from the boundary condition, we have $H_1 = c2^1 - 1 = 2c - 1 = 1$, from which we obtain $c = 1$, which in turn gives us the final result

$$H_n = 2^n - 1.$$