Definitions

Definition 1 (Subset)**.** Set A is a subset of a set B (written $A \subseteq B$) if $a \in A \implies a \in B$.

Definition 2 (Equality of sets)**.** Sets A and B are equal (written $A = B$) if $A \subseteq B \wedge B \subseteq A$.

Definition 3 (Proper subset)**.** Set A is a proper subset of B (written $A \subset B$) if $A \subseteq B \wedge A \neq B$

Definition 4 (Empty set)**.** Emptyset ($\varnothing$) is defined as a set containing no elements: $\forall x : x \notin \varnothing$.

Definition 5 (Union of sets)**.** Union of sets A and B is a set $A \cup B = \{x : x \in A \vee x \in B\}$.

Definition 6 (Intersection of sets)**.** Intersection of sets A and B is a set $A \cap B = \{x : x \in A \wedge x \in B\}$.

Definition 7 (Disjoint sets)**.** Sets A and B are disjoint if $A \cap B = \varnothing$.

Definition 8 (Set compliment)**.** Let U be the universal class, and let $A \subset U$. The compliment of A is the set $A' = \{x \in U : x \notin A\}$.

Definition 9 (Set difference)**.** The difference of sets A and B is the set $A \setminus B = A \cap B' = \{x \in A : x \notin B\}$.

Definition 10 (Cartesian product of sets)**.** Cartesian product of sets A and B is the set $A \times B = \{(a, b) : a \in A \wedge b \in B\}$.

Definition 11 (Binary Relation)**.** Relation R is a binary relation between sets A and B if $R \subseteq A \times B$.

The notation xRy means $(x, y) \in R$. Let $R \subseteq A \times B$. The set A is called the domain of R, and the set B is the co–domain of R.

Definition 12 (Endorelation)**.** Relation R is an endorelation on set A if $R \subseteq A^2 = A \times A$.

Definition 13 (Image of a set under a binary relation)**.** Let $R \subseteq A \times B$ be a binary relation. Then the image of A under R is the set $Im(R) = \{y \in B : \exists x \in A : xRy\}$.

Definition 14 (Preimage of a set under binary relation)**.** Let $R \subseteq A \times B$ be a binary relation, and let $Y \subseteq B$. Then the preimage of Y under R (written $R^{-1}(Y)$) is $R^{-1}(Y) = \{x \in A : \exists y \in Y : xRy\}$.

Definition 15 (Field of a binary relation)**.** Let R be a binary relation. Then $Field(R) = Dom(R) \cup Im(R)$.

Definition 16 (Injection)**.** A binary relation $R \subseteq A \times B$ is injective if $\forall x, z \in A, \forall y \in B : xRy \wedge zRy \implies x = z$.

In example, the relation $R = \{(x, x^2)\} \subseteq \mathbb{Z} \times \mathbb{Z}$ is not injective, since both $(2, 4)$ and $(-2, 4)$ are in R, and hence injectivity does not hold.

Definition 17 (Surjection)**.** A binary relation $R \subseteq A \times B$ is surjective if $\forall y \in B\ \exists x \in A : xRy$.

In example, the relation $R = \{(x, x^2)\} \subseteq \mathbb{Z} \times \mathbb{Z}$ is not surjective, since no preimage exists for $3 \in \mathbb{Z}$, since $\sqrt{3} \notin \mathbb{Z}$.

Definition 18 (Bijection)**.** Injective and surjective binary relation is called bijective.

Definition 19 (Set cardinality)**.** Cardinality of a set A, denoted as $|A|$, is a measure of the number of elements in the set.
$|A| = |B|$ if there exists a bijection $f : A \to B$.
$|A| \leqslant |B|$ if there exists an injection $f : A \to B$.
$|A| < |B|$ if there exists an injection $f : A \to B$, but no bijection $g : A \to B$ exists.

Definition 20 (Countable set)**.** Set A is countable if $|A| = |B|$, where $B \subseteq \mathbb{N}$.

Definition 21 (Countably infinite set)**.** Set A is countably infinite if there exists a bijection $f : A \to \mathbb{N}$.

Definition 22 (Reflexivity)**.** Binary relation R on a set A is **reflexive** if every element x in A is related to itself: $\forall x \in A : xRx$.

In example, relation $\leqslant$ on $\mathbb{Z}$ is reflexive, since $\forall a \in \mathbb{Z} : a \leqslant a$. However, the relation $<$ is not reflexive, since $a < a$ does not hold.

Definition 23 (Anti–reflexivity)**.** Binary relation R is called **anti–reflexive** if every element x in A is not related to itself: $\forall x \in A : \neg(xRx)$.

In example, relation $<$ on $\mathbb{Z}$ is anti–reflexive, since $\forall a \in \mathbb{Z} : a \nless a$.

Definition 24 (Symmetry)**.** Relation R on a set A is called **symmetric** if $\forall x, y \in A : xRy \implies yRx$.

In example, equality relation $=$ on $\mathbb{R}$ is symmetric, since $\forall a, b \in \mathbb{R} : a = b \implies b = a$.

Definition 25 (Anti–symmetry)**.** Relation R on a set A is **anti–symmetric** if $\forall x, y \in A : xRy \wedge yRx \implies x = y$.

In example, relation $\leqslant$ is anti–symmetric, since $x \leqslant y \wedge y \leqslant x \implies x = y$.

Definition 26 (Asymmetry)**.** Relation R on a set A is **asymmetric** if $\forall x, y \in A : xRy \implies \neg(yRx)$.

In example: relation $<$ on $\mathbb{R}$ is asymmetric, since $x < y \implies \neg(y < x)$.

Definition 27 (Transitivity)**.** Relation R on a set A is **transitive** if $\forall x, y, z \in A : xRy \wedge yRz \implies xRz$.

In example, it can be seen that relations $<$ and $=$ are transitive

$$a < b \wedge b < c \implies a < c\ ,$$
$$a = b \wedge b = c \implies a = c\ .$$

Definition 28 (Connexity)**.** Relation R on a set A is **connex** if $\forall x, y \in A : xRy \vee yRx$.

Definition 29 (Trichotomy)**.** R is called **trichotomous** if $\forall x, y \in A : xRy \vee yRx \vee x = y$.

Definition 30 (Left–total binary relation)**.** A binary relation $R \subseteq A \times B$ is left–total if $\forall x \in A\ \exists y \in B : xRy$.

Definition 31 (Partial function)**.** A binary relation $R \subseteq A \times B$ is a partial function if $\forall x \in A, \forall y, z \in B : xRy \wedge xRz \implies y = z$.

In example, mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by $y = \sqrt{x}$ is not a partial function, since $2 = \sqrt{4} = -2$.

Definition 32 (Function, Mapping)**.** A binary relation $R \subseteq A \times B$ is a function (or a mapping) $F : A \to B$ if it is left-total, injective, and is a partial function.

The notation $a \overset{f}{\mapsto} b$ means that element $a \in A$ is mapped to element $b \in B$ by mapping $f : A \to B$. Equivalently, the same can be expressed as $f(a) = b$. In other words, mapping $F : A \to B$ maps every element $a \in A$ to a *unique* element $b \in B$.

Definition 33 (Linear Map)**.** A **linear mapping** or **linear transformation** is a map $\mathbb{R}^n \to \mathbb{R}^m$ given by a matrix.

In example, given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,$$

we can define a map $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\forall (x, y) \in \mathbb{R}^2 : T_A(x, y) = (ax + by, cx + dy) \ .$$

This is actually matrix multiplication, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \ .$$

Definition 34 (Permutation)**.** For any set S, a bijective mapping $\pi : S \to S$ is called a **permutation**.

Suppose $S = \{1, 2, 3\}$. Define a map $\pi : S \to S$ by

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \ .$$

It is easy to verify that this map is bijective, hence this map is a permutation of S.

Definition 35 (Identity Map)**.** The **identity map** id_S is such that $\forall s \in S : s \mapsto s$.

In example, for $S = \{1, 2, 3\}$, the identity map id_S is

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \ .$$

Definition 36 (Function Composition)**.** A **composition of function** $f : A \to B$ and $g : B \to C$ is a new function $h : A \to C$ defined by

$$(g \circ f)(x) = g(f(x)) \ \ .$$

Note that $g(f(x)) = (g \circ f)(x) \neq (f \circ g)(x) = f(g(x))$.

For example, consider the following sets

$$A = \{1, 2, 3\} \qquad\qquad B = \{a, b, c\} \qquad\qquad C = \{x, y, z\} \ \ .$$

Consider maps

$$\begin{aligned} f &: A \to B \text{ defined by } \{1 \mapsto b, 2 \mapsto c, 3 \mapsto a\} \ \ , \\ g &: B \to C \text{ defined by } \{a \mapsto z, b \mapsto z, c \mapsto x\} \ \ . \end{aligned}$$

The composition $g \circ f : A \to C$ is defined by $\{1 \mapsto z, 2 \mapsto x, 3 \mapsto z\}$.

It can be seen than the composition $f \circ g$ is not a valid map.

Definition 37 (Inverse Map)**.** Let $f : A \to B$ be a function. The **inverse map** $f^{-1} : B \to A$ is a function such that

$$\begin{aligned} f \circ f^{-1} &= id_B \ \ , \\ f^{-1} \circ f &= id_A \ \ . \end{aligned}$$

Function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln(x)$ has an inverse $f^{-1}(x) = e^x$.

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) = f(e^x) = \ln e^x = x \ \ , \\ (f^{-1} \circ f)(x) &= f^{-1}(\ln x) = e^{\ln x} = x \ \ . \end{aligned}$$