ITC8190
Mathematics for Computer Science
Recap and Preparation for the Test

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Key takeaways – Sets

Definitions:

- Set: $X = \{x : x \text{ satisfies } \mathcal{P}\}$.
- Subset: $A \subseteq B \iff x \in A \implies x \in B$.
- Proper subset: $A \subset B \iff A \subset B \wedge A \neq B$.
- Equality between sets: $A = B \iff A \subseteq B \wedge B \subseteq A$.
- Disjoint sets: sets A and B are disjoint if $A \cap B = \emptyset$.
- Empty set: $\emptyset \iff \forall x : x \notin \emptyset$.
- Powerset: $\mathcal{P}(A)$ is the set of all subsets of A, including $\emptyset$ and A itself.

Key takeaways – Sets

Definitions (contd.):

- Infinite set: the set A is infinite if there exists $A' \subset A : |A'| = |A|$.
- Finite set: the set A is finite if any non-empty family of subsets has a minimal element when ordered by the inclusion relation ($\subseteq$).
- Countable set: the set A is countable if there exists an injection $f \colon A \to \mathbb{N}$.
- Countably infinite set: the set A is countebly infinite if there exists a bijection $f \colon A \to \mathbb{N}$.

Key takeaways – Sets

Set operations:

- union: $A \cup B = \{x : x \in A \vee x \in B\}$.
- intersection: $A \cap B = \{x : x \in A \wedge x \in B\}$.
- complement: $A' = \{x : x \notin A\}$.
- difference: $A \setminus B = \{x : x \in A \wedge x \notin B\}$.
- Cartesian product is the set of **ordered** pairs:
  $A \times B = \{(a, b) : a \in A, b \in B\}$.
  Cartesian product in general is not commutative:

$$A \times B \neq B \times A \quad .$$

Key takeaways – Sets

Set cardinality $|A|$ – a measure of the number of elements in the set.

- $|A| = |B|$ if there exists a bijection $f\colon A \to B$.
- $|A| \leqslant |B|$ if there exists an injection $f\colon A \to B$.
- $|A| < |B|$ if there exists an injection $f\colon A \to B$, but no bijection $g : A \to B$ exists.

Key takeaways – Binary Relations

- A binary relation R between sets A and B is any subset of the Cartesian product of $A \times B$.
- We say that $x \in A$ is related to $y \in B$ under relation R if $(a, b) \in R$, and denote it by xRy.
- The **domain** of R is the set of all $x \in A$ that are related to some $y \in B$, denoted as $Dom(R)$.

$$Dom(R) = \{x \in A : \exists y \in B : xRy\} \ .$$

- The **range** of R is the set B, denoted as $Ran(R)$.
- The **image** of A under R is the set

$$Im(R) = \{y \in B : \exists x \in A : xRy\} \ .$$

Key takeaways – Binary Relations

Binary relations possess two properties w.r.t uniqueness.

- A binary relation $R \subseteq A \times B$ is **injective** if any element in the image has a unique pre-image.

$$\forall x, z \in A, \forall y \in B : xRy \wedge zRy \implies x = z \ .$$

- A binary relation $R \subseteq A \times B$ is **functional** (or **partial function**) if for any element $x \in A$ there exists a unique element $y \in B$ such that xRy.

$$\forall x \in A, \forall y, z \in B : xRy \wedge xRz \implies y = z \ .$$

Key takeaways – Binary Relations

Binary relations possess two properties w.r.t totality.

- A binary relation $R \subseteq A \times B$ is **left-total** if every element $x \in A$ is mapped to some element $y \in B$.

$$\forall x \in A : \exists y \in B : xRy \ .$$

- A binary relation $R \subseteq A \times B$ is **surjective** if the image is equal to the range: $Im(R) = Ran(R)$. In other words,

$$\forall y \in B : \exists x \in A : xRy \ .$$

Key takeaways – Binary Relations

- A binary relation $R \subseteq A \times B$ is called a **mapping** (or a **function**) if it is left-total and functional, denoted as $R : A \to B$.
- Mappings map every element in A to a unique element in B.
- An injective mapping is an **injection**.
- Surjective mapping is a **surjection**.
- A **bijection** is an injective surjective mapping.
- Permutations are bijections.

Key takeaways – Mappings

- An **identity mapping** id is a mapping which maps every element to itself.
- Let $f\colon A \to B$ be a mapping. An **inverse mapping** $f^{-1} : B \to A$ for every given value y in the image returns a value x in the domain, such that $y = f(x)$. This value x is called a **pre-image** of y.
- The composition of a mapping with its inverse results in an identity mapping.

$$f \circ f^{-1} : B \to B \ , \qquad f^{-1} \circ f\colon A \to A \ ,$$
$$\forall y \in B : (f \circ f^{-1})(y) = y \ , \quad \forall x \in A : (f^{-1} \circ f)(x) = x \ .$$

- For a composition $f \circ g$, its inverse mapping $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, since

$$\left(f \circ g \circ g^{-1} \circ f^{-1}\right)(x) = \left(f \circ f^{-1}\right)(x) = x \ .$$

Key takeaways – Mappings

- A mapping is invertible iff it is bijective.
- If $f: A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is injective.
- If $f: A \to B$ and $g : B \to C$ are both surjective, then $g \circ f$ is surjective.
- Any permutation on a set is a bijection.
- Composition of permutations is a permutation.

Key takeaways – Endorelations

- An endorelation on a set A is a binary relation $R \subseteq A \times A$ on A.
- R is **reflexive** if any element is related to itself $\forall a \in A : aRa$.
- R is **anti–reflexive** if any element is not related to itself $\forall a \in A : \neg(aRa)$
- R is **symmetric** if $\forall a, b \in A : aRb \implies bRa$.
- R is **anti–symmetric** if

$$\forall a, b \in A : aRb \wedge bRa \implies a = b \ .$$

- R is **asymmetric** if $\forall a, b \in A : aRb \implies \neg(bRa)$.
- R is **transitive** if

$$\forall a, b, c \in A : aRb \wedge bRc \implies aRc \ .$$

Key takeaways – Endorelations

- Two elements a and b are comparable if $aRb \vee bRa$.
- R is **connex** if $\forall a, b \in A : aRb \vee bRa$. Connexity: all elements are comparable.
- R is **trichotomous** if $\forall a, b \in A : aRb \vee bRa \vee a = b$. Trichotomy: all elements are comparable or equal.
- Symmetric and transitive relation is reflexive.
- Asymmetric relation is anti–reflexive.
- Anti–reflexive and transitive relation is anti–symmetric and asymmetric.
- Anti–reflexive relation is anti–symmetric iff it is asymmetric.

Key takeaways – Equivalence relations

- Reflexive, symmetric and transitive endorelations are **equivalence relations** on a set.
- An equivalence relation $\sim$ **partitions** the underlying set X into **equivalence classes** $[x_i]$. Such a partition is called a **factor space** $X/\sim$.
- A **factor space** $X/\sim$ is an image of the set X under the equivalence relation $\sim$.
- A **partition** on a set X is a non-empty collection of subsets $X_i \subset X$ such that

$$\bigcap_i X_i = \emptyset \ , \quad \bigcup_i X_i = X \ .$$

- An **equivalence class** $[x]$ is the set

$$[x] = \{y \in X : y \sim x\} \in X/\sim \quad .$$

Key takeaways – Equivalence relations

- Two equivalence classes are either **disjoint** or **equal**.
- Any equivalence relation on a set corresponds to a partition of this set.
- Any partition of a set corresponds to an equivalence relation on this set.
- A **setoid** is a set with an equivalence relation on it.

Key takeaways – Order relations

- A (weak) **partial order** $\trianglelefteq$ on a set X is a reflexive, anti–symmetric and transitive binary relation.
- A **strict partial order** $\triangleleft$ on a set X is anti–reflexive, anti–symmetric and transitive binary relation.
- A **poset** or **partially ordered set** is a set with a partial order on it.
- Given a poset (X, R), where R is a (weak) partial order, a closed interval on this set is defined as

$$[a, b] = \{x \in X : aRx \wedge xRb\} \ .$$

- Given a poset (X, R), where R is a strict partial order, an open interval on this set is defined as

$$(a, b) = \{x \in X : aRx \wedge xRb\} \ .$$

Key takeaways – Order relations

- A **total order** (or **linear order** or a **chain**) is a connex partial order.
- A **strict total order** is a trichotomous strict partial order.
- A **totally ordered set** is a set with a total order on it.
- A **(strict) well order** is a (strict) total order in which any non–empty subset has a least element.
- A **well ordered set** is a set with a well order on it.
- $\mathbb{N}$ is a well ordered set.
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are not well ordered, but are totally ordered.
- $\mathbb{C}$ is not ordered.

Key takeaways – Extrema in a poset

- Element $m \in S \subseteq (P, R)$ is **minimal** if

$$\forall x \in S : xRm \implies x = m \ .$$

- Element $m \in S \subseteq (P, R)$ is **maximal** if

$$\forall x \in S : mRx \implies x = m \ .$$

- Element $l \in S \subseteq (P, R)$ is **least** if

$$\forall x \in S : lRx \ .$$

- Element $g \in S \subseteq (P, R)$ is **greatest** if

$$\forall x \in S : xRl \ .$$

- A poset is called **bounded** if there exist least and greatest element. Otherwise a poset is **unbounded**.

Key takeaways – Extrema in a poset

- Element $u \in (P, R)$ is an **upper bound** of $S \subseteq (P, R)$ if
$$\forall x \in S : xRu \ .$$
- Element $l \in (P, R)$ is a **lower bound** of $S \subseteq (P, R)$ if
$$\forall x \in S : lRx \ .$$
- Element $u \in (P, R)$ is **supremum** of $S \subseteq (P, R)$ (denoted as sup S) if u is the **least upper bound** of S.
- Element $l \in S \subseteq (P, R)$ is **infimum** of $S \subseteq (P, R)$ (denoted as inf S) if l is the **greatest lower bound** of S.

Key takeaways – Extrema in a poset

- If all elements are comparable (i.e. in a total order), then there exists 0 or exactly 1 minimal or maximal elements. Otherwise $0, 1$, or several minimal or maximal elements may exist.
- Greatest or least elements are unique – there may be 0 or exactly 1 greatest or least element.
- If there exists the greatest element, it is a unique maximal element. The least element (if it exists) is a unique minimal element.

Key takeaways – Extrema in a poset

- In a totally ordered set the maximal element is the greatest element.
- The greatest element in $S \subseteq (P, R)$ is one of the upper bounds, and the only upper bound that belongs to S.
- The least element in $S \subseteq (P, R)$ is one of the lower bounds, and the only lower bound that belongs to S.
- Every non-empty subset of a totally ordered set is bounded from both sides.
- The least/greatest element does not necessary exist in the set of upper/lower bounds of S.
- $\inf S$ and $\sup S$, if they exist, are unique greatest/least elements in the set of lower/upper bounds.

THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?