ITC8190
Mathematics for Computer Science
Binary Relations Between Two Sets

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The **Cartesian product** of sets A and B is the set of ordered pairs

$$A \times B = \{(a, b) : a \in A \wedge b \in B\} \ .$$

Let $A = \{x, y\}, B = \{1, 2, 3\}$. Then

$$A \times B = \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\}$$
$$B \times A = \{(1,x),(2,x),(3,x),(1,y),(2,y),(3,y)\}$$

Observe that $A \times B \neq B \times A$.

The Cartesian product of a set with itself is often denoted by

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ ,$$
$$\mathbb{Z}^n = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n \text{ times}} \ .$$

A **binary relation** R between sets A and B is the subset

$$R \subseteq A \times B : \forall x \in A, \forall y \in B : xRy \iff (x, y) \in R \quad .$$

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. An example of a relation $R \subseteq A \times B$ is a set of pairs

$$R = \{(1, a), (1, b), (2, c), (3, a)\} \quad .$$

The **domain** of $R \subseteq A \times B$ is the set

$$Dom(R) = \{x \in A : \exists y \in B : xRy\} \ .$$

The **image of** A **under** $R \subseteq A \times B$ is the set

$$Im(R) = \{y \in B : \exists x \in A : xRy\} \ .$$

The **field of** R is the set

$$Field(R) = Dom(R) \cup Im(R) \ .$$

Let

$$\begin{aligned} A &= \{1, 2, 3\} \ , \\ B &= \{a, b, c\} \ , \\ R &\subseteq A \times B \\ &= \{(1, a), (1, b), (2, c), (3, a)\} \ . \end{aligned}$$

Then

$$\begin{aligned} Dom(R) &= \{1, 2, 3\} \ , \\ Im(R) &= \{a, b, c\} \ , \\ Field(R) &= \{1, 2, 3\} \cup \{a, b, c\} \\ &= \{1, 2, 3, a, b, c\} \ . \end{aligned}$$

Let

$$\begin{aligned} A &= \{1, 2, 3, 4\} \ , \\ B &= \{a, b, c, d\} \ , \\ R &\subseteq A \times B \\ &= \{(1, a), (1, b), (3, b), (3, d)\} \ . \end{aligned}$$

Then

$$\begin{aligned} Dom(R) &= \{1, 3\} \ , \\ Im(R) &= \{a, b, d\} \ , \\ Field(R) &= \{1, 3\} \cup \{a, b, d\} \\ &= \{1, 3, a, b, d\} \ . \end{aligned}$$

A binary relation $R \subseteq A \times B$ is **injective** (or **left-unique**) if

$$\forall x, z \in A, \forall y \in B : xRy \wedge zRy \implies x = z \ .$$

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = x + 5 \in \mathbb{R}\}$$

is injective, since

$$\forall a, b \in \mathbb{R} : a + 5 = b + 5 \implies a = b \ .$$

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = x^2 \in \mathbb{R}\}$$

is not injective, since

$$\forall a, b \in \mathbb{R} : a^2 = b^2 \;\not\Longrightarrow\; a = b \;\; .$$

I.e.: $(5, 25) \in R, (-5, 25) \in R$, but $5 \neq -5$.

A binary relation $R \subseteq A \times B$ is **functional** (or **right-unique**) if

$$\forall x \in A, \forall y, z \in B : xRy \wedge xRz \implies y = z \ .$$

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = x^2 \in \mathbb{R}\}$$

is functional, since for every $x \in \mathbb{R}$ there is a unique element $x^2 \in \mathbb{R}$. The situation $xRy \wedge xRz$ is impossible.

Functional relations are also called **partial function**.

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = \sqrt{x} \in \mathbb{R}\}$$

is not functional. Because $\sqrt{25} = \pm 5$, we have $(25, 5) \in R, (25, -5) \in R$, but $5 \neq -5$.

A binary relation R is **one-to-one** if it is **injective** and **functional**. In other words, a one-to-one relation is left-unique and right-unique.

A binary relation $R \subseteq A \times B$ is **left-total** if

$$\forall x \in A \; \exists y \in B : xRy \; .$$

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = x + 5 \in \mathbb{R}\}$$

is left-total, since $\forall x \in \mathbb{R} \; \exists x + 5 \in \mathbb{R}$.

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = \sqrt{x} \in \mathbb{R}\}$$

is not left-total, since $-5 \in \mathbb{R}$, but $\sqrt{-5} \notin \mathbb{R}$.

A binary relation $R \subseteq A \times B$ is **surjective** (or **right-total**, or **onto**) if

$$\forall y \in B \; \exists x \in A : xRy \ .$$

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = x + 5 \in \mathbb{R}\}$$

is surjective, since for every $y \in \mathbb{R}$ there exists $x = y - 5 \in \mathbb{R}$.

The relation

$$R \subseteq \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y = x^2 \in \mathbb{R}\}$$

is not surjective, since $-5 \in \mathbb{R}$, but there is no $x \in \mathbb{R}$ for which $x^2 = -5$.

A binary relation is a **mapping** (or a **function**) $f\colon A \to B$ if it is functional (right-unique) and left-total.

In other words, $R \subseteq A \times B$ maps every element $a \in A$ to a *unique* element $b \in B$.

Let $f\colon A \to B$ be a mapping. We will use the following notation:

$$a \overset{f}{\mapsto} b \iff f(a) = b \ .$$

Suppose $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. The relation

$$R \subseteq A \times B = \{(1, a), (2, c), (3, a)\}$$

is a mapping, since it is functional and left-total. The relation

$$G \subseteq A \times B = \{(1, a), (1, b), (2, c), (3, c)\}$$

is not a mapping, since it is not functional – element 1 is mapped to both a and b. The relation

$$H \subseteq A \times B = \{(1, a), (2, b)\}$$

is functional, but not left-total, hence is not a mapping.

Since mapping $f\colon A \to B$ is left-total, then its **domain**

$$Dom(f) = \{x \in A : \exists y \in B : xRy\} = A \quad .$$

In other words, the domain of a mapping $f\colon A \to B$ is the set A.

The **range** of $f\colon A \to B$ is the set B.

The **image** of $f\colon A \to B$ is the set

$$f(A) = \{f(a) : a \in A\} \subseteq B \quad .$$

An **injection** is an injective mapping – a binary relation that is left-unique, right-unique, and left-total

A **surjection** (or **onto mapping**) is a surjective mapping – a binary relation that is right-unique, left-total, and right-total.

A mapping is a **bijection** (or **one-to-one correspondence**) is a mapping which is injective and surjective. In other words, left-unique, right-unique, left-total, and right-total.

Having said that, we are now ready to introduce some missing concepts from the set theory.

Cardinality of a set A (written $|A|$) is a measure of the number of elements in the set.

The sets A and B are **equinumerous** (written $|A| = |B|$), meaning that the sets A and B have the same cardinality if there exists a bijection $f \colon A \to B$.

For example, the set of even numbers $E = \{0, 2, 4, 6, \ldots\}$ has the same cardinality as the set $\mathbb{N}$, since the function $f(n) = 2n$ is a bijection $f \colon E \to \mathbb{N}$.

Cardinality of set A is **less than or equal** to the cardinality of a set B (written as $|A| \leqslant |B|$) if there exists an injective function from A to B.

Cardinality of set A is **strictly less** than the cardinality of a set B (written as $|A| < |B|$) if there exists an injective function from A to B, but no bijective function from A to B exists.

For example, the cardinality of $\mathbb{N}$ is strictly less than the cardinality of $\mathbb{R}$. The mapping $i : \mathbb{N} \to \mathbb{R}$ is injective, but it can be shown (Cantor's first uncountability proof, Cantor's diagonal argument) that there does not exist a bijection $\mathbb{N} \to \mathbb{R}$.

It is interesting to note that the cardinality of a proper subset of an infinite set is the same as the cardinality of the set itself. For instance, $\mathbb{N} \subset \mathbb{Z}$ and $|\mathbb{N}| = |\mathbb{Z}|$. Let us define a bijection $\mathbb{Z} \to \mathbb{Z}$.

$$
\begin{aligned}
&\cdots \\
-3 &\mapsto 5 \\
-2 &\mapsto 3 \\
-1 &\mapsto 1 \\
0 &\mapsto 0 \\
1 &\mapsto 2 \\
2 &\mapsto 4 \\
&\cdots
\end{aligned}
$$

A set A is **infinite** if there exists $A' \subset A$ such that $|A'| = |A|$.

A finite set A is **countable** if there exists an injective function $A \to \mathbb{N}$.

A set A is **countably infinite** if there exists a bijection $A \to \mathbb{N}$.

Infinity is the most weird, counter–intuituve, and the least understood concept in mathematics.

I.e.: an interesting phenomena involving infinite sets – the Banach–Tarski paradox.

`https://www.youtube.com/watch?v=s86-Z-CbaHA`

Theorem (Schröder–Bernstein)

If there exist injective functions $A \to B$ *and* $B \to A$*, there exists a bijection* $A \to B$*.*

Proof.

The proof is left as an exercise for the audience. □

Corollary

If $|A| \leqslant |B|$ *and* $|B| \leqslant |A|$*, then* $|A| = |B|$*.*

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THANK YOU

FOR

YOUR

ATTENTION

ANY QUESTIONS?