Exercises

Exercise 1. In a regular deck of 52 cards (two red and two black suits; 9 number and 4 picture cards in each suit), how many cards are:

(a) red?

(b) numbers?

(c) red and numbers?

(d) red or numbers (or both)?

(e) either red or numbers (but not both)?

The goal is to use the techniques from the lecture instead of direct counting.

Solution.

(a) As each rank is either a number or a picture, we can use the rule of sum to find that there are $9 + 4 = 13$ ranks in total. As there are 2 red suits with 13 ranks in each, we can use the rule of product to find that there are $2 \cdot 13 = 26$ red cards in total.

(b) As there are 4 suits with 9 numbers in each, we can use the rule of product to find that there are $4 \cdot 9 = 36$ number cards in total.

(c) As there are 2 red suits with 9 numbers in each, we can use the rule of product to find that there are $2 \cdot 9 = 18$ red numbers in total.

(d) As we already know from (a) that there are 26 red cards and from (b) that there are 36 number cards and from (c) that there are 18 red numbers, we can use the inclusion-exclusion principle to find that there are $26 + 36 - 18 = 44$ cards that are red or numbers. The subtraction is needed to adjust for the fact that the cards that are both red and numbers would otherwise be included twice (once in the red count and once more in the numbers count).

(e) As we already know from (d) that there are 44 cards that are red or numbers and from (c) that there are 18 red numbers, we can use the rule of sum in reverse to find that there are $44 - 18 = 26$ cards that are either red or numbers. The subtraction removes exactly those cards that are both red and numbers.

Exercise 2. A regular 6-sided die is thrown 5 times and the results are summed. How many ways to get an even number as the total?

Solution. Consider the situation after we have already thrown the die 4 times. If the sum of the first 4 throws is even, we need an even number on the last throw to keep te final sum even, and there are 3 ways to get that (we need 2, 4, or 6). If the sum of the first 4 throws is odd, we need an odd number on the last throw to make the final sum even, and there are 3 ways to get that also (we need 1, 3, or 5). So, irrespective of what the result of the first 4 throws was, we have 3 ways for the last throw can make the final sum even, which means there are $6 \cdot 6 \cdot 6 \cdot 6 \cdot 3 = 3\,888$ ways to get an even sum.

Exercise 3. A wedding photographer is taking pictures of a party consisting of the bride, the groom, the bride's parents and the groom's parents (six people in total). How many ways to arrange the group so that the newlyweds are:

(a) standing together?

(b) separated by at least one other person?

Solution.

(a) An easy way to compute the answer is to imagine that we first temporarily remove the bride and arrange the remaining 5 people, for which there are $P(5,5) = 5! = 120$ ways. After that we can insert the bride into each of the resulting arrangements either to the left or to the right of the groom, for a total of $120 \cdot 2 = 240$ arrangements.

(b) Following the pattern from the first case, we would still have 120 ways to arrange the 5 people except the bride, but 4 places that are not adjacent to the groom in each of the resulting arrangements, for a total of $120 \cdot 4 = 480$ ways to separate the newlyweds.

Another possibility is to note that there are $6! = 720$ ways to arrange the 6 people, and in all of them except the 120 we counted in (a), the newlyweds would be separated, so the answer could also be obtained as $720 - 240 = 480$.

Exercise 4. How many ways for a club with n members to elect an m-member board where one of the board members is designated as the president and the rest are vice-presidents?

Solution. There are multiple possible solutions, each modeling a possible way to carry out the election procedure:

- Suppose we first pick the board from the general membership, and then a president from the board. In this case we have

$$C(m,n) = \frac{n!}{m!(n-m)!}$$

ways to pick the board and in each of the boards

$$C(1,m) = \frac{m!}{1!(m-1)!} = m$$

ways to pick the president, for a total of

$$C(m,n) \cdot C(1,m) = \frac{n!}{m!(n-m)!} \cdot m = \frac{n!}{(m-1)!(n-m)!}$$

ways to complete the whole setup.

- Suppose now we first pick the president from the general membership, and then the rest of the board from the remaining members. In that scenario, there are

$$C(1,n) = \frac{n!}{1!(n-1)!} = n$$

ways to pick the president and after that

$$C(m-1, n-1) = \frac{(n-1)!}{(m-1)!((n-1)-(m-1))!} = \frac{(n-1)!}{(m-1)!(n-m)!}$$

ways to pick the $m-1$ vice-presidents from the remaining $n-1$ members, for a total of

$$C(1, n) \cdot C(m-1, n-1) = n \cdot \frac{(n-1)!}{(m-1)!(n-m)!} = \frac{n!}{(m-1)!(n-m)!}$$

ways.

- Suppose now we want to first pick the vice-presidents and then the president from the remaining members. In this (practically rather unrealistic) scenario, there would be

$$C(m-1, n) = \frac{n!}{(m-1)!(n-(m-1))!}$$

ways to pick the $m-1$ vice-presidents from the n members and then

$$C(1, n-(m-1)) = \frac{(n-(m-1))!}{1!(n-(m-1)-1)!}$$

ways to pick the president from the remaining $n-(m-1)$ members, for a total of

$$C(m-1, n) \cdot C(1, n-(m-1)) = \frac{n!}{(m-1)!(n-(m-1))!} \cdot \frac{(n-(m-1))!}{1!(n-(m-1)-1)!}$$

ways, which, after reducing the fraction by $(n-(m-1))!$ again simplifies to the same

$$\frac{n!}{(m-1)!(n-m)!}$$

we already obtained in the two previous scenarios.

As a side remark, do note that in this exercise n is the size of the base set (the general membership of the club) and m is the size of the subset (the board), in reverse of the notation we used in the lecture when defining combinations.

Exercise 5. How many ways to distribute m identical candies among n kids so that each kid gets at least one candy?

Solution. In problems like this, it's important to first be very clear about which configurations we consider distinct and which we consider equivalent. With the candies identical and the kids unique, we can fix the list of kids in some order and could then model the configurations as n-tuples $(m_1, m_2, \ldots, m_n)$ where m_1 is the number of candies given to the first kid, m_2 the number of candies given to the second kid, and so on. Obviously, the kids' shares must add up to the total number of candies we have: $m_1 + m_2 + \ldots + m_n = m$. And we also have the requirement that each kid must get at least one candy: $m_i \geq 1$ for all $1 \leq i \leq n$. Given these constraints, it would be possible to formally derive the total number of cofigurations, but it would be quite tedious.

A more efficient way to reach the solution is to model the configurations graphically:

Kid 1	Kid 2	...	Kid n
$\bullet \cdots \bullet$	$\bullet \cdots \bullet$	...	$\bullet \cdots \bullet$
m_1	m_2	...	m_n

From the figure above, it should be easy to see that each of the $n-1$ vertical lines dividing the sequence of dots representing the candies must be in a separate one of the $m-1$ spaces between the candies. If two lines would be in the same space, there would be no candies between them and then the share of the kid in that column would be zero. Thus, we have

$$C(n-1, m-1) = \frac{(m-1)!}{(n-1)!(m-n)!}$$

possible ways to distribute the candies.