

1. Apply the Euclidean algorithm and calculate

$$\gcd(26, 9)$$

$$\gcd(81, 18)$$

**Solution.**

$$\begin{aligned} \gcd(26, 9) &= \gcd(9, 8) = \gcd(8, 1) = \gcd(1, 0) = 1 \\ \gcd(81, 18) &= \gcd(18, 9) = \gcd(9, 0) = 9 \end{aligned}$$

2. Express the following pairs of numbers in the form of Bezout identity

$$\alpha a + \beta b = \gcd(a, b) .$$

$$(60, 12)$$

$$(12, 18)$$

$$(26, 9)$$

**Solution.**

$\begin{array}{r} 60 \\ 0 \end{array}$	$\begin{array}{r} 12 \\ 12 \end{array}$	$\left\  \begin{array}{r} a \\ a-5b \end{array} \right.$	$\begin{array}{r} b \\ b \end{array}$	$\left. \begin{array}{r} 12 \\ 12 \\ 0 \end{array} \right\  \left. \begin{array}{r} a \\ a \\ a-2(b-a)=3a-2b \end{array} \right.$	$\begin{array}{r} b \\ b-a \\ b-a \end{array}$
$\begin{array}{r} 26 \\ 6 \\ 6 \\ 0 \end{array}$	$\begin{array}{r} 9 \\ 9 \\ 3 \\ 3 \end{array}$	$\left\  \begin{array}{r} a \\ a-2b \\ a-2b \\ a-2b-2(3b-a) = 3a-8b \end{array} \right.$	$\begin{array}{r} b \\ b \\ b-(a-2b)=3b-a \\ 3b-a \end{array}$		

The Bezout identities are:

$$\begin{aligned} 0 \cdot 60 + 1 \cdot 12 &= \gcd(60, 12) , \\ -1 \cdot 12 + 1 \cdot 18 &= \gcd(12, 18) , \\ -1 \cdot 26 + 3 \cdot 9 &= \gcd(26, 9) . \end{aligned}$$

3. Find multiplicative modular inverse

$$2^{-1} \text{ in } \mathbb{Z}_7$$

$$4^{-1} \text{ in } \mathbb{Z}_{11}$$

$$9^{-1} \text{ in } \mathbb{Z}_{26}$$

$$2^{-1} \text{ in } \mathbb{Z}_6$$

**Solution.**

$\begin{array}{r} 2 \\ 2 \\ 0 \end{array}$	$\begin{array}{r} 7 \\ 1 \\ 1 \end{array}$	$\left\  \begin{array}{r} a \\ a \\ a-2(b-3a)=7a-2b \end{array} \right.$	$\begin{array}{r} b \\ b-3a \\ b-3a \end{array}$	$\left. \begin{array}{r} 4 \\ 4 \\ 1 \\ 1 \end{array} \right\  \left. \begin{array}{r} a \\ a \\ a-(b-2a)=3a-b \\ 3a-b \end{array} \right.$	$\begin{array}{r} b \\ b-2a \\ b-2a \\ b-2a-3(3a-b) = -11a+4b \end{array}$
$\begin{array}{r} 9 \\ 9 \\ 1 \\ 1 \end{array}$	$\begin{array}{r} 26 \\ 8 \\ 8 \\ 0 \end{array}$	$\left\  \begin{array}{r} a \\ a \\ a-(b-2a)=3a-b \\ 3a-b \end{array} \right.$	$\begin{array}{r} b \\ b-2a \\ b-2a \\ b-2a-8(3a-b)=-26a+9b \end{array}$	$\left. \begin{array}{r} 2 \\ 2 \end{array} \right\  \left. \begin{array}{r} a \\ a \end{array} \right.$	$\begin{array}{r} b \\ b-3a \end{array}$

So,

$$2^{-1} \equiv 4 \pmod{7} \quad 4^{-1} \equiv 3 \pmod{11} \quad 9^{-1} \equiv 3 \pmod{26} \quad 2^{-1} \notin \mathbb{Z}_6 .$$

4. Find additive inverse

$$-3 \text{ in } \mathbb{Z}_5$$

$$-4 \text{ in } \mathbb{Z}_{10}$$

**Solution.**

$$-3 \equiv 2 \pmod{5}$$

$$-4 \equiv 6 \pmod{10}$$

5. How many invertible elements?

$$\mathbb{Z}_6$$

$$\mathbb{Z}_6^\times$$

$$\mathbb{Z}_{11}^\times$$

**Solution.** There are 6 invertible elements in  $\mathbb{Z}_6$ , there are

$$\phi(6) = \phi(2 \cdot 3) = 6 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = 6 \cdot \frac{1}{2} \cdot \frac{2}{3} = 2 .$$

invertible elements in  $\mathbb{Z}_6^\times$ . Also, if  $n = p_1 \cdot p_2 \cdot \dots \cdot p_k$ , where  $p_i$  are primes, then  $\phi(n) = \phi(p_1) \cdot \phi(p_2) \cdot \dots \cdot \phi(p_k)$ , then

$$\phi(6) = \phi(2 \cdot 3) = \phi(2) \cdot \phi(3) = (2 - 1)(3 - 1) = 2 .$$

There are  $\phi(11) = 11 - 1 = 10$  invertible elements in  $\mathbb{Z}_{11}^\times$ .

6. Which elements have multiplicative inverses in  $\mathbb{Z}_8$  and  $\mathbb{Z}_{20}$ ?

**Solution.** In  $\mathbb{Z}_8$  : 1, 3, 5, 7. In  $\mathbb{Z}_{20}$  : 1, 3, 7, 9, 11, 13, 17, 19.

7. Write out addition and multiplication tables in  $\mathbb{Z}_5$  and  $\mathbb{Z}_8$ .

**Solution.** The Cayley tables for  $\mathbb{Z}_5$  are the following.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

The Cayley tables for  $\mathbb{Z}_8$  are the following.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3

×	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

8. Solve the following linear equations

$$\begin{array}{lll} x + 3 \equiv 2 \pmod{5} & 5 + 6 \equiv x \pmod{11} & 5x + 2 \equiv 3 \pmod{7} \\ 4x + 3 \equiv 11 \pmod{12} & x - 4 \equiv 7 \pmod{12} & 4x \equiv 2 \pmod{19} \\ 4x + 3 \equiv 5 \pmod{13} & 2x + 1 \equiv 9x - 4 \pmod{23} & 5x - 1 \equiv 3x + 1 \pmod{26} \end{array}$$

**Solution.** (a)  $x + 3 \equiv 2 \pmod{5}$ . Since  $-3 \equiv 2$  in  $\mathbb{Z}_5$ ,

$$x + 3 + 2 \equiv 2 + 2 \pmod{5} \implies x \equiv 4 \pmod{5} .$$

(b)  $5 + 6 \equiv x \pmod{11}$ . It is easy to see that  $5 + 6 = 11 \equiv 0 \pmod{11}$ .

(c)  $5x + 2 \equiv 3 \pmod{7}$ . Since  $-2 \equiv 5$  in  $\mathbb{Z}_7$ ,  $5x \equiv 1 \pmod{7}$ . Next, we need to find  $5^{-1}$  in  $\mathbb{Z}_7$  to solve the equation.

5	7	a	b
5	2	a	b-a
1	2	$a-2(b-a)=3a-2b$	b-a
1	0	$3a-2b$	$b-a-2(3a-2b) = -7a + 5b$

Therefore,  $5^{-1} = 3$ . Indeed,  $5 \cdot 3 + 2 = 17 \equiv 3 \pmod{7}$ .

(d)  $4x + 3 \equiv 11 \pmod{12}$ . Since  $-3 \equiv 9$  in  $\mathbb{Z}_{12}$ ,  $4x \equiv 8 \pmod{12}$ . There is no element  $4^{-1}$  in  $\mathbb{Z}_{12}$ , since  $\gcd(4, 12) = 4 \neq 1$ . Let us divide this equation by 4 to get  $x \equiv 2 \pmod{3}$ . This is the solution to the original equation as well. To verify, observe that  $4 \cdot 2 + 3 = 11 \equiv 11 \pmod{12}$ .

(e)  $x - 4 \equiv 7 \pmod{12}$ . Adding 4 to both sides of the equation we get  $x \equiv 11 \pmod{12}$ .

(f)  $4x \equiv 2 \pmod{19}$ . To solve the equation we need to find  $4^{-1}$  in  $\mathbb{Z}_{19}$  and multiply both sides of the equation by it.

4	19	a	b
4	3	a	b-4a
1	3	$a-(b-4a) = 5a-b$	b-4a
1	0	$5a-b$	$b-4a-3(5a-b) = -19a+4b$

So  $4^{-1} = 5$  in  $\mathbb{Z}_{19}$ . Multiplying both sides of the equation by 5, we get

$$5 \cdot 4x \equiv 5 \cdot 2 \pmod{19} \implies x \equiv 10 \pmod{19} .$$

Indeed,  $4 \cdot 10 = 40 \equiv 2 \pmod{19}$ .

(g)  $4x + 3 \equiv 5 \pmod{13}$ . Adding  $-3 \equiv 10 \in \mathbb{Z}_{13}$  to both sides of the equation, we get  $4x \equiv 2 \pmod{13}$ .

4	13	a	b
4	1	a	b-3a
0	1	$a-4(b-3a) = 13a-4b$	b-3a

So,  $4^{-1} = -3 \equiv 10 \pmod{13}$ . Multiplying both sides of the equation by 10, we get

$$10 \cdot 4x \equiv 10 \cdot 2 \pmod{13} \implies x \equiv 7 \pmod{13} .$$

Indeed,  $4 \cdot 7 + 3 = 31 \equiv 5 \pmod{13}$ .

(h)  $2x + 1 \equiv 9x - 4 \pmod{23}$ .

$$2x + 1 \equiv 9x - 4 \pmod{23} \implies 16x + 1 \equiv -4 \pmod{23} \implies 16x \equiv 18 \pmod{23} .$$

16	23		a	b
16	7		a	b-a
2	7		$a-2(b-a)=3a-2b$	b-a
2	1		$3a-2b$	$b-a-3(3a-2b) = -10a+7b$
0	1		$3a-2b-2(-10a+7b) = 23a-16b$	$-10a+7b$

Therefore,  $16 \cdot 13 \cdot x \equiv 18 \cdot 13 \pmod{23} \implies x \equiv 4 \pmod{23}$ . Indeed,  $2 \cdot 4 + 1 \equiv 9 \cdot 4 - 4 \pmod{23} \implies 9 \equiv 32 \pmod{23}$ .

(i)  $5x - 1 \equiv 3x + 1 \pmod{26}$ .

$$\begin{aligned} 5x - 1 \equiv 3x + 1 \pmod{26} &\implies 5x \equiv 3x + 2 \pmod{26} \\ &\implies 2x \equiv 2 \pmod{26} \implies x \equiv 1 \pmod{26} . \end{aligned}$$

Indeed,  $5 \cdot 1 - 1 \equiv 3 \cdot 1 + 1 \pmod{26}$ .

9. Solve the systems of linear equations

$\begin{cases} a + b \equiv 17 \pmod{26} \\ 2a + b \equiv 0 \pmod{26} \end{cases}$	$\begin{cases} a + b \equiv 17 \pmod{26} \\ 4a + b \equiv 1 \pmod{26} \end{cases}$
$\begin{cases} a + b \equiv 17 \pmod{26} \\ 3a + b \equiv 0 \pmod{26} \end{cases}$	$\begin{cases} 5a + b \equiv 21 \pmod{26} \\ 16a + b \equiv 10 \pmod{26} \end{cases}$
$\begin{cases} 8a + b \equiv 8 \pmod{26} \\ 5a + b \equiv 13 \pmod{26} \end{cases}$	

**Solution.** (a)  $\begin{cases} a + b \equiv 17 \pmod{26} \\ 2a + b \equiv 0 \pmod{26} \end{cases}$  .

Subtracting the first equation from the second, we get  $a \equiv 9 \pmod{26}$ . Substituting this value of  $a$  into the first equation, we have  $b + 9 \equiv 17 \implies b \equiv 8 \pmod{26}$ . To verify, observe that  $9 + 8 \equiv 17 \pmod{26}$  and  $2 \cdot 9 + 8 \equiv 0 \pmod{26}$ .

(b)  $\begin{cases} a + b \equiv 17 \pmod{26} \\ 4a + b \equiv 1 \pmod{26} \end{cases}$  .

Subtracting the second equation from the first one, we get  $23a \equiv 16 \pmod{26}$ . Next, we find  $23^{-1}$  in  $\mathbb{Z}_{26}$ .

23	26		a		b
23	3		a		b-a
2	3		$a-7(b-a)=8a-7b$		b-a
2	1		$8a-7b$		$b-a-(8a-7b) = -9a+8b$
0	1		$8a-7b-2(-9a+8b) = 26a-23b$		$-9a+8b$

Therefore,  $23^{-1} = 17$  in  $\mathbb{Z}_{26}$ . Multiplying both sides of the equation by 17, we have

$$17 \cdot 23a \equiv 17 \cdot 16 \implies a \equiv 12 \pmod{26} .$$

Substituting  $a$  into the first equation, we have  $b + 12 \equiv 17 \implies b \equiv 5 \pmod{26}$ . To verify, observe that  $12 + 5 = 17 \pmod{26}$  and  $4 \cdot 12 + 5 = 53 \equiv 1 \pmod{26}$ .

$$(c) \begin{cases} a + b \equiv 17 \pmod{26} \\ 3a + b \equiv 0 \pmod{26} \end{cases} .$$

Subtracting the second equation from the first one, we get  $24a \equiv 17 \pmod{26}$ . This equation is not solvable, since there is no element  $24^{-1}$  in  $\mathbb{Z}_{26}$  and  $2 \nmid 17$ .

$$(d) \begin{cases} 5a + b \equiv 21 \pmod{26} \\ 16a + b \equiv 10 \pmod{26} \end{cases} .$$

Subtracting the second equation from the first one, we get  $15a \equiv 11 \pmod{26}$ . Next we look for  $15^{-1}$  in  $\mathbb{Z}_{26}$ .

15	26		a		b
15	11		a		b-a
4	11		$a-(b-a) = 2a-b$		b-a
4	3		$2a-b$		$b-a-2(2a-b)=-5a+3b$
1	3		$2a-b-(-5a+3b)=7a-4b$		$-5a+3b$
1	0		$7a-4b$		$-5a+3b-3(7a-4b) = -26a + 15b$

Therefore,  $15^{-1} = 7$  in  $\mathbb{Z}_{26}$ . We have  $a \equiv 7 \cdot 11 \equiv 25 \pmod{26}$ . Substituting the value of  $a$  into the first equation, we get  $b = 21 - 5 \cdot 25 \equiv 0 \pmod{26}$ .

$$(e) \begin{cases} 8a + b \equiv 8 \pmod{26} \\ 5a + b \equiv 13 \pmod{26} \end{cases} .$$

Subtracting the second equation from the first one, we get

$$3a \equiv 21 \pmod{26} \implies a \equiv 7 \pmod{26} .$$

Substituting the value of  $a$  into the first equation, we get

$$7 \cdot 8 + b \equiv 8 \pmod{26} \implies 4 + b \equiv 8 \pmod{26} \implies b \equiv 4 \pmod{26} .$$

To verify that the solution is indeed correct, observe that  $8 \cdot 7 + 4 = 60 \equiv 8 \pmod{26}$  and  $5 \cdot 7 + 4 = 39 \equiv 13 \pmod{26}$ .