ITC8190
Mathematics for Computer Science
Counting: Basic Methods

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Combinatorics is an area of mathematics concerned with

- existence
- construction
- optimization
- counting

of various finite structures.

Today we will focus on counting.

Obvious brute force solution:

- construct all instances, then count them one by one.

Tuple is a fixed-length ordered sequence of elements of a given m-element base set M.

Two tuples are considered the same if they have equal lengths and equal elements in corresponding positions.

An n-element tuple is called n-tuple.

For example, for the base set $M = \{1, 2, 3\}$, we have the following 9 distinct 2-tuples (pairs):
$(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$, $(3, 3)$.

Permutation is a tuple with the additional requirement that all elements must be distinct.

For example, for the base set $M = \{1, 2, 3\}$, we have the following 6 distinct 2-permutations:
$(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 3)$, $(3, 1)$, $(3, 2)$.

An m-permutation of m-element base set M is called just a permutation of M.

Combination is a fixed-size subset of the base set.

Two combinations are considered the same if they are equal as sets, i.e. they consist of the same elements, irrespective of the order in which the elements are listed.

For example, for the base set $M = \{1, 2, 3\}$, we have the following 3 distinct 2-combinations:
$\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$.

Partition is a division of the base set M into a set of non-empty subsets in such a way that each element of M belongs to exactly one subset.

Two partitions are considered the same if they are equal as sets (of sets).

For example, for the base set $M = \{1, 2, 3\}$, we have the following 5 distinct partitions:
"partition" into one part: $\{\{1, 2, 3\}\}$,
into two parts: $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$,
into three parts: $\{\{1\}, \{2\}, \{3\}\}$.

Rule of product: if there's w_1 ways of doing a first step and w_2 ways of doing a second step, then there are a total of $w = w_1 \cdot w_2$ ways of doing those two steps.

The total number of n-tuples of m-element set:

$$T(n, m) = \underbrace{m \cdot m \cdot \ldots \cdot m}_{n} = m^n.$$

The number of n-permutations of m-element set:

$$T(n, m) = m \cdot (m-1) \cdot \ldots \cdot (m-(n-1)) = \frac{m!}{(m-n)!} = (m)_n.$$

The rule of product can only be directly used if the number of possibilities for the second step is the same for all ways of doing the first step. This is a problem for applying the rule for counting n-combinations.

But, the rule of product can also be used in reverse: from each n-combination we can generate $n!$ distinct n-permutations, and this gives us each n-permutation exactly once; therefore the number of n-combinations:

$$C(n, m) = \frac{P(n, m)}{n!} = \frac{m!}{n! \cdot (m-n)!} = \binom{m}{n}.$$

Rule of sum: if there's w_1 ways of doing one step and w_2 ways of doing another step, and we must do one or the other, but not both, then there are a total of $w = w_1 + w_2$ ways of doing one of those two steps.

Recurrent formula for n-combinations of m-element set:

$$C(n,m) = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = m \\ C(m-1, n-1) + C(m-1, n) & \text{if } 0 < n < m. \end{cases}$$

The rule of sum can be used twice to compute the number of partitions of an m-element set:

$$B(m) = \sum_{n=1}^{m} S(n, m),$$

where $S(n, m)$ is the number of partitions into n parts:

$$S(n,m) = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = m \\ n \cdot S(n, m-1) + S(n-1, m-1) & \text{if } 1 < n < m. \end{cases}$$

Like the rule of product, the rule of sum can also be reversed.

For example, to count the integers in $1 \ldots 100$ not divisible by 3, it is easier to count the number of integers that are divisible by 3 (which is 33) and then conclude that the remaining $100 - 33 = 67$ must be not divisible.

Inclusion-exclusion principle is a generalization of the rule of sum for overlapping sets.

For just two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For example, to count the integers in $1 \ldots 100$ divisible by 3 or 5 (or both), we need to observe that

- the number of those divisible by 3 is 33,
- the number of those divisible by 5 is 20,
- the number of those divisible by both 3 and 5, i.e. the number of those divisible by $\mathrm{lcm}(3, 5) = 15$ is 6,

and therefore the answer is $33 + 20 - 6 = 47$.

Inclusion-exclusion principle in general case:

$$|\bigcup_{i\in\{1,\ldots,n\}} A_i| = \Sigma_1 - \Sigma_2 + \ldots + (-1)^{n+1}\Sigma_n,$$

where

$$\Sigma_i = \sum_{\{j_1,\ldots,j_i\}\subseteq\{1,\ldots,n\}} |A_{j_1} \cap \ldots \cap A_{j_i}|.$$

This is quite inconvenient to use manually for larger n, though, as the number of summands in Σ_i is $\binom{n}{i}$ and the total number of summands over all Σ_i is 2^n.

Linear recurrences

Many counting problems yield recurrent equations.

While there's no general method to solve all recurrences, there are specific methods for common classes, for example linear recurrences.

General linear recurrence (of k-th order) has the form
$A_0 = m_0$, $A_1 = m_1$, $A_{k-1} = m_{k-1}$,
$A_{n+k} = b_1 A_{n+k-1} + b_2 A_{n+k-2} + ... + b_k A_n + f(n)$,
where m_i and b_j are constants and $f(n)$ is an arbitrary function on n.

Linear homogeneous recurrence of first order

Let's first consider the special case
$A_0 = m_0,$
$A_{n+1} = b_1 A_{n-1}.$

Here it is easy to see the general solution
$A_n = m_0 b_1^n.$

Linear homogeneous recurrence of second order

$A_0 = m_0$, $A_1 = m_1$,
$A_{n+2} = b_1 A_{n-1} + b_2 A_{n-2}$.

By analogy with the previous case, we'll start from looking for solution of the form $A_n = q^n$. Substituting into the recurrence, we get

$$q^{n+2} = b_1 q^{n+1} + b_2 q^n$$

$$q^n (q^2 - b_1 q - b_2) = 0$$

Now the characteristic equation $q^2 - b_1 q - b_2 = 0$ may have
(a) two distinct solutions $q_1 \neq q_2$, or
(b) two coinciding solutions $q_1 = q_2$.

(a) In case of two distinct solutions $q_1 \neq q_2$, we can see that any linear combination

$$A_n = c_1 q_1^n + c_2 q_2^n$$

also satisfies the general recurrent equation, and we can use the boundary conditions to derive the equations

$$\begin{cases} c_1 + c_2 & = m_0 \\ c_1 q_1 + c_2 q_2 & = m_1 \end{cases}$$

for finding the c_1 and c_2 to get the particular solution that satisfies also the boundary conditions.

(b) In case of coinciding solutions $q_1 = q_2$, we can see that any linear combination

$$A_n = c_1 q_1^n + c_2 n q_1^n$$

also satisfies the general recurrent equation, and we can use the boundary conditions to derive the equations

$$\begin{cases} c_1 & = m_0 \\ c_1 q_1 + c_2 q_1 & = m_1 \end{cases}$$

for finding the c_1 and c_2 to get the particular solution that satisfies also the boundary conditions.

Linear recurrences in general case can be solved following the same pattern:

- first we extract the (k-th degree) characteristic equation from the recurrent rule and solve it;
- if the characteristic equation has solutions $q_1, \ldots, q_s$ with multiplicities $k_1, \ldots, k_s$ (where $k_1 + \ldots + k_s = k$), the general solution has the form

$$\begin{aligned} A_n = & (c_{1,0} + c_{1,1}n + \ldots + c_{1,k_1-1}n^{k_1-1})q_1^n + \\ & (c_{2,0} + c_{2,1}n + \ldots + c_{2,k_2-1}n^{k_2-1})q_2^n + \\ & \ldots + \\ & (c_{s,0} + c_{s,1}n + \ldots + c_{s,k_s-1}n^{k_s-1})q_s^n, \end{aligned}$$

  where the values of the multipliers $c_{i,j}$ can be found using the boundary conditions.

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THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?