

ITC8190  
Mathematics for Computer Science  
Equivalence Relations on Sets

Aleksandr Lenin

October 2nd, 2018

Relation  $R$  on a set  $A$  is called an **equivalence relation** iff  $R$  is reflexive, symmetric, and transitive.

Let us verify if  $=$  is an equivalence relation on  $\mathbb{N}$ .

Reflexivity: any element  $a$  is equal to itself ( $a = a$ ).

Symmetry: if  $a = b$  then also  $b = a$ .

Transitivity: if  $a = b$  and  $b = c$ , then also  $a = c$ .

Hence,  $=$  is an equivalence relation on  $\mathbb{N}$ .

$$R \subseteq \mathbb{N} \times \mathbb{N} = \{(0, 0), (1, 1), (2, 2), \dots\} .$$

Suppose that  $f$  and  $g$  are differentiable functions on  $\mathbb{R}$ . Let  $\sim$  be an equivalence relation defined by

$$f(x) \sim g(x) \iff \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} .$$

It is clear that  $\sim$  is reflexive and symmetric.

To show transitivity, suppose  $f(x) \sim g(x)$  and  $g(x) \sim h(x)$ . The condition  $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}$  is satisfied if  $f(x)$  and  $g(x)$  differ by a constant.

$$f(x) - g(x) = c_1 \quad ,$$

$$g(x) - h(x) = c_2 \quad ,$$

$$f(x) - h(x) = f(x) - g(x) + g(x) - h(x) = c_1 + c_2 \quad .$$

This implies  $f(x) \sim h(x)$ .

An equivalence relation gives rise to a partition via equivalence classes.

Picture will be drawn on the whiteboard.

Such a partition is called a **factor space**, and the following notation is used  $X/\sim$ , where  $X$  is the underlying set, and  $\sim$  is the equivalence relation.

A set with an equivalence relation on it is called a **setoid**.

A **partition**  $P$  on a set  $X$  is a collection of non-empty subsets  $X_1, X_2, \dots$  such that they are all disjoint, meaning that  $X_i \cap X_j = \emptyset$  for  $i \neq j$ , and  $\bigcup_k X_k = X$ .

Let  $\sim$  be an equivalence relation on a set  $X$  and let  $x \in X$ . Then the **equivalence class**  $[x] \in X / \sim$  is

$$[x] = \{y \in X : y \sim x\} .$$

## Lemma 1

*Given an equivalence relation  $\sim$  on a set  $X$ , there exists at least one non-empty equivalence class.*

### Proof.

Suppose there exists an equivalence relation  $\sim$  on  $X$ , and let  $x \in X$  is non-empty. By reflexivity of  $\sim$ ,  $x \sim x$ , and so  $x \in [x]$ . Hence, the equivalence class  $[x]$  is non-empty.  $\square$

## Theorem 1

*Given an equivalence relation  $\sim$  on a set  $X$ , the equivalence classes of  $X$  form a partition of  $X$ .*

### Proof.

Suppose there exists an equivalence relation  $\sim$  on  $X$ . We need to show that  $\sim$  forms a partition of  $X$ . By Lemma 1,

$\bigcup_{x \in X} [x] = X$ . Let  $x, y \in X$ . We will show that either  $[x] \cap [y] = \emptyset$  or  $[x] = [y]$ . Suppose  $[x] \cap [y]$  is non-empty.

$$z = [x] \cap [y] \neq \emptyset \implies z \sim x \wedge z \sim y \implies x \sim y \implies [x] \subseteq [y] .$$

Similarly,  $y \sim x \implies [y] \subseteq [x]$ , and so  $[x] = [y]$ . Therefore, two equivalence classes are disjoint or exactly the same.  $\square$

## Theorem 2

If  $P = \{X_i\}$  is a partition of a set  $X$ , then there is an equivalence relation on  $X$  with equivalence classes  $X_i$ .

### Proof.

Let  $P = \{X_i\}$  be a partition of a set  $X$ . Let  $a \sim b \iff a \in X_i \wedge b \in X_i$ . Clearly,  $\sim$  is reflexive. To show symmetry, observe that

$$x \sim y \implies x \in X_i \wedge y \in X_i \implies y \sim x .$$

For transitivity, observe that

$$x \sim y \wedge y \sim z \implies x \in X_i \wedge y \in X_i \wedge z \in X_i \implies x \sim z .$$

Clearly,  $\sim$  is an equivalence relation on  $X$ . □



## Corollary 1

*Any two equivalence classes are either disjoint or equal.*

## Corollary 2

*Every equivalence relation on a set corresponds to a partition of this set.*

## Corollary 3

*Any partition of a set corresponds to an equivalence relation which gives rise to this partition.*

In example,

$$\mathbb{Z}/\sim: a \sim b \iff a \equiv b \pmod{2}$$

contains two equivalence classes  $[0]$  and  $[1]$  – even and odd numbers.

$$\begin{aligned} [0] &= \{\dots, -4, -2, 0, 2, 4, \dots\} , \\ [1] &= \{\dots, -3, -1, 1, 3, 5, \dots\} . \end{aligned}$$

It can be seen that  $[0] \cap [1] = \emptyset$  and  $[0] \cup [1] = \mathbb{Z}$ .

Equivalence classes in  $\mathbb{Z}/\sim: a \sim b \iff a \equiv b \pmod{3}$ :

$$\begin{aligned} [0] &= \{\dots, -3, 0, 3, 6, \dots\} , \\ [1] &= \{\dots, -2, 1, 4, 7, \dots\} , \\ [2] &= \{\dots, -1, 2, 5, 8, \dots\} , \end{aligned}$$

form another partition of  $\mathbb{Z}$ , since  $[0] \cap [1] \cap [2] = \emptyset$  and  $[0] \cup [1] \cup [2] = \mathbb{Z}$ .

The set of integers  $\mathbb{Z}$  is an image of  $\mathbb{N} \times \mathbb{N}$ , under  $\sim$ .

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim , \quad (a, b) \sim (c, d) \iff a - b = c - d .$$

The set of rational numbers is an image of the set  $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$  under  $\sim$ .

$$\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim , \quad (a, b) \sim (c, d) \iff ac = bd .$$

The set  $\mathbb{Z}_n$  is an image of  $\mathbb{Z}$  under congruence relation.

$$\begin{aligned} \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\} = \mathbb{Z} / \sim : a \sim b &\iff a \equiv b \pmod{n} \\ &\iff (a - b) | n . \end{aligned}$$



THANK YOU  
FOR  
YOUR  
ATTENTION  
ANY QUESTIONS?