ITC8190
Mathematics for Computer Science
Mathematical Induction

Aleksandr Lenin

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Suppose we wish to show that for all $n \in \mathbb{N}$:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

- easy to verify for small values such as $n = 1, 2, 3, 4$
- impossible to verify for all $n \in \mathbb{N}$ on a case-by-case basis.

To prove the formula in general, a more generic proof method is required.

This method of proof is known as **mathematical induction**.

Instead of attempting to verify a statement on a case-by-case basis, a **specific proof for the smallest considered integer** is given, followed by a **generic argument** showing that **if the statement holds for a given case, it must also hold for the next case in the sequence**.

I.e., suppose we want to show that we can climb as high as we like on a ladder.

So show this using mathematical induction, we show that

- We can climb on the first rung (**the basis**)
- From each rung we can climb on the next one (**the step**)

Concrete Mathematics, page 3 margin

Suppose we wish to show that for all $n \in \mathbb{N}$:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

The formula is true for 1, since $1 = \frac{1(1+1)}{2}$. If it holds for some n, we show that it holds for $n+1$.

$$\begin{aligned} 1 + 2 + \ldots + n + (n+1) &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)[(n+1)+1]}{2} \end{aligned}$$

is exactly the formula for $(n+1)$th case.

We summarize mathematical induction in the following axiom.

Definition 1 (First Principle of Mathematical Induction)

Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer n_0. If for all integers $k \geqslant n_0 : S(k) \implies S(k+1)$, then $S(n)$ is true for all integers $n \geqslant n_0$.

Let us show that for all $n \geqslant 3$ it holds that $2^n > n + 4$.

Base The statement is true for $n_0 = 3$, since

$$8 = 2^3 \geqslant 3 + 4 = 7 \ .$$

Step Assume $2^k > k + 4$ some $k \geqslant 3$. Then for $k + 1$

$$2^{k+1} = 2{\cdot}2^k \geqslant 2(k{+}4) = 2k{+}8 > k{+}5 = (k{+}1){+}4 \ .$$

Therefore, $2^{k+1} \geqslant (k + 1) + 4$.

Every integer $10^{n+1} + 3 \cdot 10^n + 5$ is divisible by 9 for $n \in \mathbb{N}$.

The statement is true for $n = 1$, since

$$10^2 + 30 + 5 = 135 = 9 \cdot 15 \ .$$

Suppose $10^{k+1} + 3 \cdot 10^k + 5$ is divisible for some $k \geqslant 1$. Then

$$\begin{aligned} 10^{k+2} + 3 \cdot 10^{k+1} + 5 &= 10 \cdot 10^{k+1} + 10 \cdot 3 \cdot 10^k + 50 - 45 \\ &= 10 \cdot (10^{k+1} + 3 \cdot 10^k + 5) - 45 \end{aligned}$$

is divisible by 9.

We will prove the binomial theorem using mathematical induction.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad ,$$

where $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad .$$

For $n = 1$ the binomial theorem is easy to verify.

$$(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = a^0 b^1 + a^1 b_0 = a + b \quad .$$

Lemma 1

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Proof.

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k} \ .\end{aligned}$$

□

Assume the binomial theorem holds for $n \geqslant 1$, then

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\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b)\left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] a^k b^{n+1-k} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k} \quad .
\end{aligned}
$$

Definition 2 (Second Principle of Mathematical Induction)

Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$, and suppose $S(n_0)$ holds for some integer n_0. If for $k \geqslant n_0$

$$S(n_0), S(n_0+1), \ldots, S(k) \implies S(k+1) \quad ,$$

then $S(n)$ is true for all $n \geqslant n_0$.

The Principle of Mathematical Induction is equivalent to the Principle of Well–Ordering.

Definition 3 (Principle of Well–Ordering)

Every non-empty subset of $\mathbb{N}$ has a least element.

The set $\mathbb{Z}$ is not well-ordered, since it does not contain a smallest element.

Lemma 2

The Principle of Mathematical Induction implies that 0 *is the least natural number.*

Proof.

Let $S = \{n \in \mathbb{N} : n \geqslant 0\}$. Then $0 \in S$. Now assume that $n \in S$, and $n \geqslant 0$. Since $n + 1 \geqslant 0$, then $n + 1 \in S$. Hence, by induction, every natural number is greater than or equal to 0. □

Theorem 1

The Principle of Mathematical Induction implies that the natural numbers are well ordered.

Proof.

We must show that if $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, then S contains a smallest element. If $0 \in S$, then the theorem is true by Lemma 2. Assume that if $k \in S$ with $0 \leqslant k \leqslant n$, then S contains a smallest element. We will show that if $k \in S$ with $0 \leqslant k \leqslant n+1$, then S has a smallest element. If S does not contain an integer less than $n+1$, then $n+1$ is the smallest integer in S. Otherwise, since S is non-empty, S must contain an integer less than or equal to n. In this case, by induction, S contains a smallest integer. $\square$

Induction can also be useful in formulating definitions. For instance, there are two ways to define the factorial of a positive integer n.

- The explicit definition: $n! = 2 \cdot 3 \cdots (n-1) \cdot n$.
- The inductive or recursive definition: $1! = 1$ and $n! = n(n-1)$ for $n > 1$.

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THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?