1. Alice and Bob generate a session key using the Diffie-Hellman key establishment protocol. They agree on a finite cyclic group $\mathbb{Z}_{23}^{\times}$ generated by 5. What is the order of $\mathbb{Z}_{23}^{\times}$? Suppose that Alice's private exponent is 2, and Bob's private exponent is 3, what is the session key generated by Alice and Bob?

   Solution. Alice computes $5^2 = 25 \equiv 2 \pmod{23}$ and sends it to Bob. Bob computes $5^3 = 125 \equiv 10 \pmod{23}$ and sends this value to Alice. To get the session key, Alice computes $10^2 = 100 \equiv 8 \pmod{23}$ and Bob computes $2^3 = 8 \pmod{23}$.

2. Consider the following key agreement protocol between Alice (A) and Bob (B). Prior to starting any communication, Alice and Bob generate their secret keys ω_A and ω_B. Alice generates the session key K. To share K with Bob, the following sequence of messages is executed.

   (1) Alice $\to$ Bob: $\omega_A \oplus K$.

   (2) Bob $\to$ Alice: $\omega_B \oplus \omega_A \oplus K$

   (3) Alice $\to$ Bob: $\omega_A \oplus \omega_B \oplus \omega_A \oplus K = \omega_B \oplus K$

   After receiving the last message, Bob computes $\omega_B \oplus \omega_B \oplus K = K$. At this point Alice and Bob have the shared key K which they use to encrypt the communication. Can adversary Carol obtain the key K by eavesdropping on the communication channel?

   Solution. Having obtained messages $1, 2$ and 3, Carol can run an exclusive or operation on all three messages and reveal the shared secret K. Observe that

$$m_1 \oplus m_2 = \omega_A \oplus K \oplus \omega_B \oplus \omega_A \oplus K = \omega_B \;\; ,$$
$$(m_1 \oplus m_2) \oplus m_3 = \omega_B \oplus \omega_B \oplus K = K \;\; .$$

3. Provide prime factorization of the following integers:

(a) 64	(b) 120
(c) 375	(d) 47

   Solution.

(a) $64 = 2^6$	(b) $120 = 2 \cdot 3 \cdot 4 \cdot 5$
(c) $375 = 15 \cdot 25 = 3 \cdot 5^3$	(d) 47

4. Given a list of functions in asymptotic notation, order them by growth rate (slowest to fastest).

(a) $\Theta(n \log_2 n)$	(b) $\Theta(n^2)$	(c) $\Theta(n)$	(d) $\Theta(1)$	(e) $\Theta(2^n)$
(f) $\Theta(n^3)$	(g) $\Theta(n!)$	(h) $\Theta(\log_2 n)$	(i) $\Theta(n^2 \log_2 n)$	(j) $\Theta(2^n \log^2 n)$

   Solution. (a) $\Theta(1)$

   (b) $\Theta(n^2 \log_2 n)$

   (c) $\Theta(n)$

(d) $\Theta(n \log_2 n)$

(e) $\Theta(n^2)$

(f) $\Theta(n^2 \log_2 n)$

(g) $\Theta(n^3)$

(h) $\Theta(2^n)$

(i) $\Theta(2^n \log^2 n)$

(j) $\Theta(n!)$

5. Check if the following conditions are true

$$\begin{aligned} (a) &\quad \Theta(n+30) = \Theta(3n-1) \ , \\ (b) &\quad \Theta(n^2+2n-10) = \Theta(n^2+3n) \ , \\ (c) &\quad \Theta(n^3 \cdot 3n) = \Theta(n^2+3n) \ . \end{aligned}$$

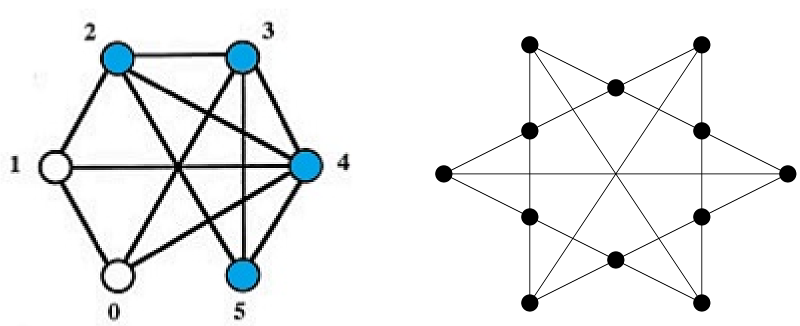
Solution. (a) true (b) true (c) false

6. Write each of the following functions in O notation.

$$(a) \quad 5 + 0.001n^3 + 0.025n \qquad (b) \quad 500n + 100n^{1.5} \qquad (c) \quad 0.3n + 5n^{1.5} + 2.5n^{1.75}$$

Solution.

$$O(n^3) \ , \qquad\qquad O(n^{1.5}) \ , \qquad\qquad O(n^{1.75}) \ .$$



(a) Maximal clique problem (b) graph 3-coloring problem

7. Find the maximal clique in the graph shown in Fig. 1a. A subgraph H of a graph G is a maximal clique in G if there is an edge between every pair of vertices in H, and there is no vertex in $G \setminus H$ connected to every vertex in H.

Solution. The vertices belonging to the maximal clique are marked in blue.

8. Provide a 3–coloring of the graph shown in Fig. 1b so that any two adjacent vertices do not share the same color.

Solution. To verify if the graph in Fig. 1b is 3–colorable, we reduce this problem to a 3–SAT instance, and run it through an SMT solver. The reduction of the 3–SAT to 3–colorability is pretty straightforward and can be easily inferred by readin the 3–SAT model. You can see the 3–SAT formulation of the task in file named ”3sat”. To verify uncolorability, copy-paste the model into Z3 solver the online version of which can be found here: `https://rise4fun.com/z3`.

The result is that this graph is **usatisfiable**, which means that it is also not 3–colorable. The unsatisfiability core (the least set of edges, the vertices of which cannot be 3-colorable) is

```
(v1v2 v2v3 v3v4 v2v4 v4v5 v5v6 v4v6 v6v7 v7v8 v6v8 v8v9 v9v10
v8v10 v10v11 v11v12 v10v12 v1v12 v2v12 v1v7 v5v11 v3v9)
```