1 Exercises

Exercise 1. How would you test if an integer n is prime, using the brute-force approach?

Solution. We need to check if n is co-prime to all integers in the range $[2, \sqrt{n}]$. If it is the case, then n is prime.

Exercise 2. Apply Fermat primality test to verify the primality of $351, 511$, and 717.

Solution. For 351, take $a = 2$ and we have $2^{350} \bmod 351 = 121 \neq 1$, hence, 2 is a Fermat witness, and 351 is composite.

For 511, take $a = 2$ and we have $2^{510} \bmod 511 = 64 \neq 1$, hence 2 is a Fermat witness, and 511 is composite.

For 717, take $a = 2$ and we have $2^{716} \bmod 717 = 4 \neq 1$, hence 2 is a Fermat witness, and 717 is composite.

Exercise 3. Solve for x: $x^2 \bmod 63 = 1$, where 63 is the product of two primes 7 and 9.

Solution. We need to find the square roots of 1 is $\mathbb{Z}_{63}$. By the Chinese Remainder Theorem, $\mathbb{Z}/63\mathbb{Z} \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$, and isomorphism $\varphi : \mathbb{Z}/63\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ is given by $\varphi(n \bmod 63) \mapsto (x \bmod 7, x \bmod 9) = (x_7, x_9)$. To find the square roots of unity, we need to find elements (x_7, x_9) that would satisfy $(x_7^2 \bmod 7, x_9^2 \bmod 9) = (1, 1)$. For this, we solve a system of equations $\begin{cases} x_7^2 \bmod 7 = 1 \ , & \text{and hence } x_7{=}1 \text{ or } x_7 = 6 \ , \\ x_9^2 \bmod 9 = 1 \ , & \text{and hence } x_9{=}1 \text{ or } x_9 = 8 \ . \end{cases}$ The possible values for (x_7, x_9) form 4 combinations: $(1, 1), (1, 8), (6, 1), (6, 8)$. Now we need to map these elements back into $\mathbb{Z}/63\mathbb{Z}$. The inverse isomorphism $\psi : \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/63\mathbb{Z}$ is given by the solution to the CRT $\begin{cases} x \equiv x_p \pmod p \\ x \equiv x_q \pmod q \end{cases}$ given by $\psi : (x_p, x_q) \mapsto \beta q x_p + \alpha p x_q \bmod n$, where α and β are the Bézout coefficients of $\gcd(x_p, x_q) = \alpha p + \beta q$. The Bézout identity is $\gcd(7, 9) = 4 \cdot 7 + (-3) \cdot 9 = 1$. Hence, $\alpha = 4$ and $\beta = -3$.

Solving $\begin{cases} x \equiv 1 \pmod 7 \\ x \equiv 1 \pmod 9 \end{cases}$ has a trivial solution $x = 1 \in \mathbb{Z}/63\mathbb{Z}$.

Solving $\begin{cases} x \equiv 1 \pmod 7 \\ x \equiv 8 \pmod 9 \end{cases}$ gives the solution $x = 8 \in \mathbb{Z}/63\mathbb{Z}$.

Solving $\begin{cases} x \equiv 6 \pmod 7 \\ x \equiv 1 \pmod 9 \end{cases}$ gives the solution $x = -3 \cdot 9 \cdot 6 + 4 \cdot 7 \cdot 1 = -134 \equiv 55 \in \mathbb{Z}/63\mathbb{Z}$.

Solving $\begin{cases} x \equiv 6 \pmod 7 \\ x \equiv 8 \pmod 9 \end{cases}$ gives the solution $x = -3 \cdot 9 \cdot 6 + 4 \cdot 7 \cdot 8 = 62 \in \mathbb{Z}/63\mathbb{Z}$.

To conclude, there are four 2-nd roots of unity in $\mathbb{Z}/63\mathbb{Z}$, they are: $1, 8, 55, 62$.

Exercise 4. Adversary Carol has intercepted two RSA cryptograms, $y_1 = 537$ sent by Alice to Bob, and $y_2 = 285$ sent by Alice to Eve. The public key of Bob is $(e_1 = 18, n_1 = 943)$, and the public key of Eve is $(e_2 = 19, n_2 = 943)$. What is the message m sent by Alice to Bob and Eve?

Solution. Carol knows that Bob and Eve receive the same message m. Hence,

$$537 = m^{18} \bmod 943 \ ,$$
$$285 = m^{19} \bmod 943 \ .$$

Since the public exponents of Bob and Eve are co-prime, meaning that $\gcd(18, 19) = 1$, the Bézout identity applies.

$$\gcd(18, 19) = 1 \cdot 19 + (-1) \cdot 18 = 1 \ .$$

Carol needs to combine y_1 and y_2 in such a way that would exploit the Bézout identity to get m. Observe that given $\alpha \cdot e_1 + \beta \cdot e_2 = 1$ implies that

$$y_1^{\alpha} \cdot y_2^{\beta} = (m^{e_1})^{\alpha} \cdot (m^{e_2})^{\beta} = m^{\alpha \cdot e_1} \cdot m^{\beta \cdot e_2} = m^{\alpha \cdot e_1 + \beta \cdot e_2} = m^1 = m \ .$$

Given that $\alpha = -1$, and hence $y_1^{\alpha} = 537^{-1} \equiv 72 \pmod{943}$ and $\beta = 1$, Carol needs to compute

$$m = 72 \cdot 285 \bmod 943 = 717 \ .$$

Exercise 5. Suppose that adversary Carol has intercepted three cryptograms y_1, y_2, y_3 sent to three different users whose public keys are $(e, n_1), (e, n_2), (e, n_3)$. Notice that all public keys use the same public exponent, and different moduli. What does Carol needs to do to reconstruct the message m?

Solution. Carol knows that

$$y_1 = m^3 \bmod n_1$$
$$y_2 = m^3 \bmod n_2$$
$$y_3 = m^3 \bmod n_3$$

and she knows that $m < n_1, m < n_2, m < n_3$. Hence, $m^3 < n_1 \cdot n_2 \cdot n_3$, and therefore m^3 is the solution to the CRT

$$\begin{cases} x \bmod n_1 = y_1 \\ x \bmod n_2 = y_2 \\ x \bmod n_3 = y_3 \end{cases}$$

CRT guarantees the existence of the solution, and that the solution is unique. Since m^3 solves the CRT, it must be this unique solution. All that Carol needs to do now is to calculate

$$m = \sqrt[3]{m^3}$$

to reconstruct the message m.

If the public exponent is relatively small, say, n, then the adversary needs to intercept n cryptograms in order to be able to decipher it.