Exercise 1. Prove that for all $n \in \mathbb{N}$

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \ .$$

Solution. It holds for $n = 1$, since $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$. Suppose it holds for some n. Then for $n+1$,

$$\begin{aligned} 1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} = \frac{(n+1)(2n^2+7n+6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \ . \end{aligned}$$

Exercise 2. Prove that for all $n \in \mathbb{N}$

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \ .$$

Solution. It holds for $n = 1$, since $1^3 = \frac{1^2(1+1)^2}{4}$. Suppose it holds for some n. Then for $n+1$,

$$\begin{aligned} 1^3 + 2^3 + \ldots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^2(n+1) = \frac{n^2(n+1)^2 + 4(n+1)^2(n+1)}{4} \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4} \ . \end{aligned}$$

Exercise 3. Prove that $n! > 2^n$ for $n \geqslant 4$.

Solution. It holds for $n = 4$, since $4! = 24 > 2^4 = 16$. Suppose it holds for some n. Then for $n+1$,

$$(n+1)! = n! \cdot (n+1) > 2 \cdot n! > 2 \cdot 2^n > 2^{n+1} \ .$$

Exercise 4. Prove that for all $n \in \mathbb{N}$,

$$x + 4x + 7x + \ldots + (3n-2)x = \frac{n(3n-1)x}{2} \ .$$

Solution. It holds for $n = 1$, since $x = \frac{2x}{2}$. Suppose it holds for some n. Then for $n+1$,

$$\begin{aligned} x + 4x + \ldots + (3n-2)x + (3(n+1)-2)x &= \frac{n(3n-1)x}{2} + (3n+1)x = \frac{n(3n-1)x + 2(3n-1)x + 4x}{2} \\ &= \frac{x(3n^2+5n+2)}{2} = \frac{(n+1)(3n+2)x}{2} \ . \end{aligned}$$

Exercise 5. Prove that for all $n \in \mathbb{N}$, $10^{n+1} + 10^n + 1$ is divisible by 3.

Solution. It holds for $n = 1$, since $10^2 + 10 + 1 = 111 = 3 \cdot 37$. Suppose it holds for some n. Then for $n+1$,

$$10^{n+2} + 10^{n+1} + 1 = 10 \cdot 10^{n+1} + 10 \cdot 10^n + 10 - 9 = 10 \cdot (10^{n+1} + 10^n + 1) - 9 \ .$$

Exercise 6. Prove that for all $n \in N$, $4 \cdot 10^{2n} + 9 \cdot 10^{2n-1} + 5$ is divisible by 99.

Solution. It holds for $n = 1$, since $4 \cdot 10^2 + 9 \cdot 10 + 5 = 495 = 99 * 5$. Suppose it holds for some n, then for $n + 1$,

$$\begin{aligned} 4 \cdot 10^{2(n+1)} + 9 \cdot 10^{2(n+1)-1} + 5 &= 10^2 \cdot 4 \cdot 10^{2n} + 10^2 \cdot 9 \cdot 10^{2n-1} + 500 - 495 \\ &= 10^2 \cdot (4 \cdot 10^{2n} + 9 \cdot 10^{2n-1} + 5) - 495 \ . \end{aligned}$$

Exercise 7. Prove that for all $n \in \mathbb{N}$

$$1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1 \ .$$

Solution. It holds for $n = 1$, since $2^{1+1} - 1 = 3 = 1 + 2$. Suppose it holds for some n. Then for $n + 1$,

$$1 + 2 + 2^2 + \ldots + 2^n + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1} = 2^{n+2} - 1 \ .$$

Exercise 8. Prove that for all $n \in \mathbb{N}$

$$\frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \ .$$

Solution. It clearly holds for $n = 1$. Suppose it holds for some n, then for $n + 1$,

$$\begin{aligned} \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2} \ . \end{aligned}$$

Exercise 9. Prove that $2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$ for all $n \geqslant 1$.

Solution. For $n = 1$ we have $2^1 = 2^2 - 2$. Suppose it holds for some n. For $n + 1$ we have:

$$2^1 + 2^2 + \ldots + 2^n + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1} = 2 \cdot 2^{n+1} - 2 = 2^{(n+1)+1} - 2 = 2^{n+2} - 2 \ .$$

Exercise 10. Prove that $1 + 2 + 3 + \ldots n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Solution. It holds for $n = 1$, since $1 = \frac{1 \cdot (1+1)}{2}$. Suppose that it holds for some n. Then, for $n + 1$

$$1 + 2 + 3 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \ .$$

Exercise 11. Prove that for all $n \in \mathbb{N}, n \geqslant 1$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} \ .$$

Solution. It holds for $n = 1$, since $\frac{1}{1 \cdot 2} = 1 - \frac{1}{1+1} = \frac{1}{2}$. Suppose it holds for some n. For $n + 1$ we have:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+2} \ .$$