Exercise 1. Let A be a set. Show that $A \cup A = A$.

Solution.

$$A \cup A = \{x : x \in A \vee x \in A\} = \{x \in A\} = A \ .$$

Exercise 2. Let A be a set. Show that $A \setminus A = \emptyset$.

Solution.

$$A \setminus A = A \cap A' = \{x : x \in A \wedge x \notin A\} = \emptyset \ .$$

Exercise 3. Let A be a set. Show that $A \cap \emptyset = \emptyset$.

Solution.

$$A \cap \emptyset = \{x : x \in A \wedge x \in \emptyset\} = \emptyset \ .$$

Exercise 4. Let A, B, C be sets. Show that $A \cap (B \cap C) = (A \cap B) \cap C$.

Solution.

$$\begin{aligned} A \cap (B \cap C) &= A \cap \{x : x \in B \wedge x \in C\} \\ &= \{x : x \in A \wedge x \in B \wedge x \in C\} \\ &= \{x : x \in A \wedge x \in B\} \cap C \\ &= (A \cap B) \cap C \ . \end{aligned}$$

Exercise 5. Let A, B be sets. Show that $A \cap B = B \cap A$.

Solution.

$$A \cap B = \{x : x \in A \wedge x \in B\} = B \cap A \ .$$

Exercise 6. Let A, B, C be sets. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

$$\begin{aligned} A \cap (B \cup C) &= A \cap \{x : x \in B \vee x \in C\} \\ &= \{x \in A \wedge (x \in B \vee x \in C)\} \\ &= \{(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\ &= (A \cap B) \cup (A \cap C) \ . \end{aligned}$$

Exercise 7. Let A, B be sets. Show that $(A \cap B)' = A' \cup B'$.

Solution. We must show that $(A \cap B)' \subseteq A' \cup B'$ and $(A \cap B)' \supseteq A' \cup B'$.

$$\begin{aligned} &x \in (A \cap B)' \implies x \notin (A \cap B) \implies x \notin A \vee x \notin B \implies x \in A' \vee x \in B' \implies x \in A' \cup B' \ . \\ &x \in A' \cup B' \implies x \in A' \vee x \in B' \implies x \notin A \vee x \notin B \implies x \notin A \cap B \implies x \in (A \cap B)' \ . \end{aligned}$$

Therefore, $(A \cap B)' \subseteq A' \cup B'$ and $(A \cap B)' \supseteq A' \cup B'$. Hence, $(A \cap B)' = A' \cup B'$.

Exercise 8. Let A, B be sets. Show that $A \subseteq B$ iff $A \cap B = A$.

Solution. First, we show that $A = A \cap B \implies A \subseteq B$.

$$x \in A \cap B \implies x \in A \wedge x \in B \implies (x \in A \implies x \in B) \implies A \subseteq B \ .$$

Next, we show that $A \subseteq B \implies A = A \cap B$. For this, we first show that $A \subseteq B \implies A \subseteq A \cap B$.

$$\begin{aligned} A \subseteq B &\implies (x \in A \implies x \in B) \implies (x \in A \implies x \in A \wedge x \in B) \\ &\implies (x \in A \implies x \in A \cap B) \implies A \subseteq A \cap B \ . \end{aligned}$$

Finally, it is easy to see that $A \cap B \subseteq A$.

$$x \in A \cap B \implies x \in A \wedge x \in B \implies x \in A \implies A \cap B \subseteq A \ .$$

Exercise 9. Let A, B be sets. Show that $(A \setminus B) \cap (B \setminus A) = \emptyset$.

Solution.

$$(A \setminus B) \cap (B \setminus A) = (A \cap B') \cap (B \cap A') = (A \cap A' \cap B \cap B') = \emptyset \ .$$

Exercise 10. Let A, B, C be sets. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Solution.

$$\begin{aligned} (A \cup B) \times C &= \{(a, b) : (a \in A \vee a \in B) \wedge b \in C\} \\ &= \{(a, b) : (a \in A \wedge b \in C) \vee (a \in B \wedge b \in C)\} \\ &= (A \times C) \cup (B \times C) \ . \end{aligned}$$

Exercise 11. Let A, B be sets. Show that $(A \cap B) \setminus B = \emptyset$.

Solution.

$$(A \cap B) \setminus B = A \cap B \cap B' = \emptyset \ .$$

Exercise 12. Let A, B be sets. Show that $(A \cup B) \setminus B = A \setminus B$.

Solution.

$$\begin{aligned} (A \cup B) \setminus B = (A \cup B) \cap B' &= (A \cap B') \cup (B \cap B') \\ &= (A \cap B') \cup \emptyset = (A \cap B') = A \setminus B \ . \end{aligned}$$

Exercise 13. Let A, B be sets. Show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Solution.

$$\begin{aligned} A \setminus (B \cup C) = A \cap (B \cup C)' &= A \cap (B' \cap C') \\ &= (A \cap B') \cap (A \cap C') = (A \setminus B) \cap (A \setminus C) \ . \end{aligned}$$

Exercise 14. Let A, B be sets. Show that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

Solution.

$$\begin{aligned}(A\cap B)\setminus(A\cap C) &= (A\cap B)\cap(A\cap C)' = (A\cap B)\cap(A'\cup C')\\ &= (A\cap B\cap A')\cup(A\cap B\cap C') = A\cap B\cap C' = A\cap(B\setminus C)\ .\end{aligned}$$

Exercise 15. Let A, B be sets. Show that $(A\setminus B)\cup(B\setminus A) = (A\cup B)\setminus(A\cap B)$.

Solution.

$$\begin{aligned}(A\setminus B)\cup(B\setminus A) &= (A\cap B')\cup(B\cap A') = (A\cup B)\cap(A\cup A')\cap(B'\cup B)\cap(B'\cup A')\\ &= (A\cup B)\cap(A'\cup B') = (A\cup B)\cap(A\cap B)' = (A\cup B)\setminus(A\cap B)\ .\end{aligned}$$