Theorem 1 (Korselt, 1899)**.** A positive composite integer n is a Carmichael number iff

- n is square-free – n is divisible by no perfect square other than 1
- for all prime factors p of n it holds that $p - 1 | n - 1$

From the theorem it follows that

1. Carmichael numbers are odd
2. Carmichael numbers are cyclic – n and $\varphi(n)$ are co-prime
3. Carmichael numbers have at least 3 positive prime factors.

To see that the smallest Carmichael number $561 = 3 \cdot 11 \cdot 17$ satisfies the Korselt's criterion, observe that 561 is square-free and $2|560, 10|560$ and $16|560$.

In 1939, J. Chernick proved a theorem showing that the number $(6k+1)(12k+1)(18k+1)$ is a Carmichael number if the three factors are all prime. In example, for $k = 1$ we have $7 \cdot 13 \cdot 19 = 1729$ which is a Carmichael number.

How can we distinguish Carmichael numbers?

No Carmichael number is either an Euler-Jacobi pseudoprime or a strong pseudoprime to every base relatively prime to it. In theory, either the Euler-Jacobi test, or a strong primality test (i.e., Miller-Rabin) will prove the compositeness of the considered Carmichael number.

An odd integer n is called an Euler–Jacobi probable prime to base a if $\gcd(a, n) = 1$ and

$$a^{(n-1)/2} \equiv \left(\frac{a}{n}\right) \pmod{n} ,$$

where $\left(\frac{a}{b}\right)$ is the Jacobi symbol. If n is an odd composite integer that satisfies the congruence, then it is called Euler–Jacobi pseudoprime. Solovay and Strassen have shown that for every composite n, at least $n/2$ bases less than n, n is not an Euler-Jacobi pseudoprime.

For example, 561 is an Euler–Jacobi probable prime to base 50,

$$50^{280} \bmod 561 = 1 = \left(\frac{50}{561}\right) ,$$

as well as to base 2

$$2^{280} \bmod 561 = 1 = \left(\frac{2}{561}\right) ,$$

but not for base 11,

$$11^{280} \bmod 561 = 220 \neq \left(\frac{12}{561}\right) .$$

Therefore, we have an evidence of the compositeness of 561.

Exercise 1. Verify that the following Carmichael numbers satisfy the Korselt's criterion:

$$\begin{aligned} 1105 &= 5 \cdot 13 \cdot 17 & \qquad 1729 &= 7 \cdot 13 \cdot 19 \\ 2465 &= 5 \cdot 17 \cdot 29 & \qquad 2821 &= 7 \cdot 13 \cdot 31 \\ 6601 &= 7 \cdot 23 \cdot 41 & \qquad 8911 &= 7 \cdot 19 \cdot 67 \end{aligned}$$

Exercise 2. Test the following numbers for primality using Euler-Jacobi primality test.

1105	1729	2465	2821	6601	8911

Exercise 3. Apply the Miller-Rabin test and check if the following integers are strong probable primes.

1105	1729	2465	2821	6601	8911