Exercise 1. Show that the set $\mathbb{Z}$ is countably infinite.

Solution. We need to demonstrate the existence of a bijection $\mathbb{Z} \to \mathbb{N}$. Indeed, such a mapping exists:

$$\begin{aligned}
&\dots \\
-3 &\mapsto 5 \\
-2 &\mapsto 3 \\
-1 &\mapsto 1 \\
0 &\mapsto 0 \\
1 &\mapsto 2 \\
2 &\mapsto 4 \\
3 &\mapsto 6 \\
&\dots
\end{aligned}$$

In other words, 0 maps to itself, negative numbers are mapped to odd numbers using function $g(n) = -(2n+1)$, positive numbers are mapped to even numbers using the function $f(n) = 2n$.

Exercise 2. Given a permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

on a set $S = \{1, 2, 3\}$, define an inverse permutation π^{-1}.

Solution.

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \;, \qquad \pi^{-1} \circ \pi = \pi \circ \pi^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \;.$$

Exercise 3. Let $f(x) = x^2$ and $g(x) = 2x + 5$. Define compositions $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they the same?

Solution.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = f(2x+5) = (2x+5)^2 \;, \\
(g \circ f)(x) &= g(f(x)) = g(x^2) = 2x^2 + 5 \;.
\end{aligned}$$

The compositions of mappings are not the same, since $f \circ g \neq g \circ f$.

Exercise 4. Let $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$. Define compositions $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they the same?

Solution.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x \;, \\
(g \circ f)(x) &= g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x \;.
\end{aligned}$$

The compositions of mappings are the same, meaning that $f \circ g = g \circ f$.

Exercise 5. Let $h : S \to T$ be a bijection, and let h^{-1} be its inverse. What are the mappings $h \circ h^{-1}$ and $h^{-1} \circ h$?

Solution.

$$h \circ h^{-1} : T \to T = id_T \ ,$$
$$h^{-1} \circ h : S \to S = id_S \ .$$

Exercise 6. Let A and B be sets. Let $A' \subset A$ and $I : A' \to A'$. Is the composition $f \circ I$ the same as the restriction of f to A'?

Definition 1 (Restriction of a mapping to a subset)**.** If $f : A \to B$ is a mapping and $A' \subset A$, the mapping $f' : A' \to B$ given by $x \mapsto f(x)$ for $x \in A'$ is called the restriction of f to A'.

Solution. We will show that the composition $f \circ I$ behaves like $f : A' \to B$ defined by $x \mapsto f(x)$. Indeed, since I is the identity map, $I(x) = x$, and so

$$\forall x \in A' : (f \circ I)(x) = f(I(x)) = f(x) \ .$$

Exercise 7. Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective, show that f is injective.

Solution. For any two values a, b and any function h, it holds that $a = b \implies h(a) = h(b)$. To show that f is injective, we assume the condition $f(a) = f(b)$. Then

$$f(a) = f(b) \implies g(f(a)) = g(f(b)) \implies (g \circ f)(a) = (g \circ f)(b) \ .$$

By injectivity of $g \circ f$, it holds that

$$(g \circ f)(a) = (g \circ f)(b) \implies a = b \ .$$

And so, we have shown that f is injective:

$$f(a) = f(b) \implies (g \circ f)(a) = (g \circ f)(b) \implies a = b \ .$$

Exercise 8. Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective and f is surjective, show that g is injective.

Solution. We need to show that g is injective: $\forall a, b \in B : g(a) = g(b) \implies a = b$. So we start by assuming that $g(a) = g(b)$. By surjectivity of f, there exist elements $x, y \in A$ such that $f(x) = a$ and $f(y) = b$. And so we have

$$g(a) = g(b) \implies g(f(x)) = g(f(y)) \implies (g \circ f)(x) = (g \circ f)(y) \ .$$

By injectivity of $g \circ f$,

$$(g \circ f)(x) = (g \circ f)(y) \implies x = y \ ,$$

and if we apply f to both sides,

$$x = y \implies f(x) = f(y) \implies a = b \ .$$

Therefore, by showing that

$$g(a) = g(b) \implies (g \circ f)(x) = (g \circ f)(y) \implies x = y \implies f(x) = f(y) \implies a = b \ ,$$

we have shown that g is injective.

Exercise 9. Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is surjective and g is injective, show that f is surjective.

Solution. To show that f is surjective, we need to show that for any element $b \in B$ there exists an element $a \in A$ such that $f(a) = b$. Let $b \in B$. By injectivity of g, there exists a unique $c \in C$ such that $c = g(b)$. By surjectivity of $g \circ f$, there exists $a \in A$ such that $c = (g \circ f)(a) = g(f(a))$. By injectivity of g, $g(f(a)) = g(b) \implies f(a) = b$. Therefore, we have shown for any $b \in B$ there exists $a \in A$ such that $f(a) = b$, and hence f is surjective.

Exercise 10. Let A and B be sets. Let $f : A \to B$ and $g : B \to A$ be mappings. If $f \circ g = id_B$, show that f is surjective.

Solution. We need to show that f is surjective, meaning that we need to show that for every $b \in B$ there exists $a \in A$ such that $b = f(a)$. The composition $f \circ g$ is an identity mapping, meaning that for every $b \in B$, it holds that $(f \circ g)(b) = f(g(b)) = b$. We can see that if we take $a = g(b)$, then indeed, for every $b \in B$, there exists $a = g(b) \in A$ such that $f(a) = f(g(b)) = (f \circ g)(b) = b$.

Exercise 11. Let A and B be sets. Let $f : A \to B$ and $g : B \to A$ be mappings. If $g \circ f = id_A$, show that f is injective.

Solution. We need to show that f is injective, meaning that for any $a, b \in A : f(a) = f(b) \implies a = b$. We start by assuming $f(a) = f(b)$. If we apply g to both sides, since g is a function, the result will not change, meaning that

$$f(a) = f(b) \implies g(f(a)) = g(f(b)) \implies (g \circ f)(a) = (g \circ f)(b) \ .$$

We know that $g \circ f$ is an identity map, meaning that for every $a \in A$, it holds that $(g \circ f)(a) = a$. In turn, it means that

$$(g \circ f)(a) = (g \circ f)(b) \implies a = b \ .$$

We have shown that $f(a) = f(b) \implies a = b$, and therefore f is injective.