

ITC8190
Mathematics for Computer Science
Equivalence Relations on Sets

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Relation R on a set A is called an **equivalence relation** iff R is reflexive, symmetric, and transitive.

Let us verify if $=$ is an equivalence relation on \mathbb{N} .

Reflexivity: any element a is equal to itself ($a = a$).

Symmetry: if $a = b$ then also $b = a$.

Transitivity: if $a = b$ and $b = c$, then also $a = c$.

Hence, $=$ is an equivalence relation on \mathbb{N} .

$$R \subseteq \mathbb{N} \times \mathbb{N} = \{(0, 0), (1, 1), (2, 2), \dots\} .$$

Suppose that f and g are differentiable functions on \mathbb{R} . Let \sim be an equivalence relation defined by

$$f(x) \sim g(x) \iff \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} .$$

It is clear that \sim is reflexive and symmetric.

To show transitivity, suppose $f(x) \sim g(x)$ and $g(x) \sim h(x)$. The condition $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}$ is satisfied if $f(x)$ and $g(x)$ differ by a constant.

$$f(x) - g(x) = c_1 \quad ,$$

$$g(x) - h(x) = c_2 \quad ,$$

$$f(x) - h(x) = f(x) - g(x) + g(x) - h(x) = c_1 + c_2 \quad .$$

This implies $f(x) \sim h(x)$.

An equivalence relation gives rise to a partition via equivalence classes.

Picture will be drawn on the whiteboard.

Such a partition is called a **factor space**, and the following notation is used X/\sim , where X is the underlying set, and \sim is the equivalence relation.

A set with an equivalence relation on it is called a **setoid**.

A **partition** P on a set X is a collection of non-empty subsets X_1, X_2, \dots such that they are all disjoint, meaning that $X_i \cap X_j = \emptyset$ for $i \neq j$, and $\bigcup_k X_k = X$.

Let \sim be an equivalence relation on a set X and let $x \in X$. Then the **equivalence class** $[x] \in X / \sim$ is

$$[x] = \{y \in X : y \sim x\} .$$

Lemma 1

Given an equivalence relation \sim on a set X , there exists at least one non-empty equivalence class.

Proof.

Suppose there exists an equivalence relation \sim on X , and let $x \in X$. By reflexivity of \sim , $x \sim x$, and so $x \in [x]$. Hence, the equivalence class $[x]$ is non-empty. \square

Theorem 1

Given an equivalence relation \sim on a set X , the equivalence classes of X form a partition of X .

Proof.

Suppose there exists an equivalence relation \sim on X . We need to show that \sim forms a partition of X . By Lemma 1,

$\bigcup_{x \in X} [x] = X$. Let $x, y \in X$. We will show that either $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Suppose $[x] \cap [y]$ is non-empty.

$$z \in [x] \cap [y] \neq \emptyset \implies z \sim x \wedge z \sim y \implies x \sim y \implies [x] \subseteq [y] .$$

Similarly, $y \sim x \implies [y] \subseteq [x]$, and so $[x] = [y]$. Therefore, two equivalence classes are disjoint or exactly the same. \square

Theorem 2

If $P = \{X_i\}$ is a partition of a set X , then there is an equivalence relation on X with equivalence classes X_i .

Proof.

Let $P = \{X_i\}$ be a partition of a set X . Let $a \sim b \iff a \in X_i \wedge b \in X_i$. Clearly, \sim is reflexive. To show symmetry, observe that

$$x \sim y \implies x \in X_i \wedge y \in X_i \implies y \sim x .$$

For transitivity, observe that

$$x \sim y \wedge y \sim z \implies x \in X_i \wedge y \in X_i \wedge z \in X_i \implies x \sim z .$$

Clearly, \sim is an equivalence relation on X . □

Corollary 1

Any two equivalence classes are either disjoint or equal.

Corollary 2

Every equivalence relation on a set corresponds to a partition of this set.

Corollary 3

Any partition of a set corresponds to an equivalence relation which gives rise to this partition.

In example,

$$\mathbb{Z}/\sim: a \sim b \iff a \equiv b \pmod{2}$$

contains two equivalence classes $[0]$ and $[1]$ – even and odd numbers.

$$\begin{aligned} [0] &= \{\dots, -4, -2, 0, 2, 4, \dots\} , \\ [1] &= \{\dots, -3, -1, 1, 3, 5, \dots\} . \end{aligned}$$

It can be seen that $[0] \cap [1] = \emptyset$ and $[0] \cup [1] = \mathbb{Z}$.

Equivalence classes in $\mathbb{Z}/\sim: a \sim b \iff a \equiv b \pmod{3}$:

$$\begin{aligned} [0] &= \{\dots, -3, 0, 3, 6, \dots\} , \\ [1] &= \{\dots, -2, 1, 4, 7, \dots\} , \\ [2] &= \{\dots, -1, 2, 5, 8, \dots\} , \end{aligned}$$

form another partition of \mathbb{Z} , since

$$[0] \cap [1] = [1] \cap [2] = [0] \cap [2] = \emptyset \text{ and } [0] \cup [1] \cup [2] = \mathbb{Z}.$$

The set of integers \mathbb{Z} is an image of $\mathbb{N} \times \mathbb{N}$, under \sim .

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim , \quad (a, b) \sim (c, d) \iff a - b = c - d .$$

The set of rational numbers is an image of the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ under \sim .

$$\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim , \quad (a, b) \sim (c, d) \iff ac = bd .$$

The set \mathbb{Z}_n is an image of \mathbb{Z} under the congruence relation.

$$\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\} = \mathbb{Z} / \equiv : a \equiv b \pmod{n} \iff n | (a - b) .$$



THANK YOU
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ANY QUESTIONS?