# ITC8190
# Mathematics for Computer Science
## Mathematical Induction

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Suppose we wish to show that for all $n \in \mathbb{N}$:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

- easy to verify for small values such as $n = 1, 2, 3, 4$
- impossible to verify for all $n \in \mathbb{N}$ on a case-by-case basis.

To prove the formula in general, a more generic proof method is required.

This method of proof is known as **mathematical induction**.

Instead of attempting to verify a statement on a case-by-case basis, a **specific proof for the smallest considered integer** is given, followed by a **generic argument** showing that **if the statement holds for a given case, it must also hold for the next case in the sequence**.

I.e., suppose we want to show that we can climb as high as we like on a ladder.

So show this using mathematical induction, we show that

- We can climb on the first rung (**the basis**)
- From each rung we can climb on the next one (**the step**)

*Concrete Mathematics, page 3 margin*

Suppose we wish to show that for all $n \in \mathbb{N}$:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

The formula is true for 1, since $1 = \frac{1(1+1)}{2}$. If it holds for some $n$, we show that it holds for $n+1$.

$$\begin{aligned} 1 + 2 + \ldots + n + (n+1) &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)[(n+1)+1]}{2} \end{aligned}$$

is exactly the formula for $(n+1)$th case.

We summarize mathematical induction in the following axiom.

## Definition 1 (First Principle of Mathematical Induction)

Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer $n_0$. If for all integers $k \geqslant n_0 : S(k) \implies S(k+1)$, then $S(n)$ is true for all integers $n \geqslant n_0$.

Let us show that for all $n \geqslant 3$ it holds that $2^n > n + 4$.

Base The statement is true for $n_0 = 3$, since

$$8 = 2^3 > 3 + 4 = 7 \ .$$

Step Assume $2^k > k + 4$ some $k \geqslant 3$. Then for $k + 1$

$$2^{k+1} = 2{\cdot}2^k > 2(k{+}4) = 2k{+}8 > k{+}5 = (k{+}1){+}4 \ .$$

Therefore, $2^{k+1} > (k + 1) + 4$.

Every integer $10^{n+1} + 3 \cdot 10^n + 5$ is divisible by 9 for $n \in \mathbb{N}$.

The statement is true for $n = 1$, since

$$10^2 + 30 + 5 = 135 = 9 \cdot 15 \quad .$$

Suppose $10^{k+1} + 3 \cdot 10^k + 5$ is divisible for some $k \geqslant 1$. Then

$$\begin{aligned} 10^{k+2} + 3 \cdot 10^{k+1} + 5 &= 10 \cdot 10^{k+1} + 10 \cdot 3 \cdot 10^k + 50 - 45 \\ &= 10 \cdot (10^{k+1} + 3 \cdot 10^k + 5) - 45 \end{aligned}$$

is divisible by 9.

We will prove the binomial theorem using mathematical induction.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad ,$$

where $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad .$$

For $n = 1$ the binomial theorem is easy to verify.

$$(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = a^0 b^1 + a^1 b^0 = a + b \quad .$$

## Lemma 1

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

## Proof.

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k} \; .\end{aligned}$$

□

Assume the binomial theorem holds for $n \geqslant 1$, then

$$
\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b)\left(\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k} \\
&= a^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}a^k b^{n+1-k} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n+1-k} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]a^k b^{n+1-k} + b^{n+1} \\
&= a^{n+1} + \sum_{k=0}^{n}\binom{n+1}{k}a^k b^{n+1-k} = \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k} \quad .
\end{aligned}
$$

## Definition 2 (Second Principle of Mathematical Induction)

Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$, and suppose $S(n_0)$ holds for some integer $n_0$. If for $k \geqslant n_0$

$$S(n_0), S(n_0+1), \ldots, S(k) \implies S(k+1) \ ,$$

then $S(n)$ is true for all $n \geqslant n_0$.

The Principle of Mathematical Induction is equivalent to the Principle of Well–Ordering.

### Definition 3 (Principle of Well–Ordering)

Every non-empty subset of $\mathbb{N}$ has a least element.

The set $\mathbb{Z}$ is not well-ordered, since it does not contain a smallest element.

## Lemma 2

*The Principle of Mathematical Induction implies that* $0$ *is the least natural number.*

## Proof.

Let $S = \{n \in \mathbb{N} : n \geqslant 0\}$. Then $0 \in S$. Now assume that $n \in S$, and $n \geqslant 0$. Since $n + 1 \geqslant 0$, then $n + 1 \in S$. Hence, by induction, every natural number is greater than or equal to 0. $\square$

## Theorem 1

*The Principle of Mathematical Induction implies that the natural numbers are well ordered.*

## Proof.

We must show that if $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, then $S$ contains a smallest element. If $0 \in S$, then the theorem is true by Lemma 2. Assume that if $k \in S$ with $0 \leqslant k \leqslant n$, then $S$ contains a smallest element. We will show that if $k \in S$ with $0 \leqslant k \leqslant n+1$, then $S$ has a smallest element. If $S$ does not contain an integer less than $n+1$, then $n+1$ is the smallest integer in $S$. Otherwise, since $S$ is non-empty, $S$ must contain an integer less than or equal to $n$. In this case, by induction, $S$ contains a smallest integer. $\square$

Induction can also be useful in formulating definitions. For instance, there are two ways to define the factorial of a positive integer $n$.

- The explicit definition: $n! = 2 \cdot 3 \cdots (n-1) \cdot n$.
- The inductive or recursive definition: $1! = 1$ and $n! = n(n-1)!$ for $n > 1$.

# THANK YOU
# FOR
YOUR
# ATTENTION
# ANY QUESTIONS?