ITC8190
Mathematics for Computer Science
Counting: Solving Recurrences

Ahto Truu

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Rabbits

Consider a farmer starting to keep rabbits:

- in month 0, he has no rabbits;
- in month 1, he buys a pair of newborn rabbits;
- in each following month:
  - all existing rabbits continue to live;
  - each pair of rabbits at least two months old produces a new pair of rabbits as offspring.

How many pairs of rabbits will the farmer have in month n?

Fibonacci numbers

The number of pairs of rabbits is defined by:
$F_0 = 0,$
$F_1 = 1,$
$F_{n+2} = F_{n+1} + F_n$ (for $n \geq 0$).

So, we have
$F_2 = F_1 + F_0 = 1 + 0 = 1,$
$F_3 = F_2 + F_1 = 1 + 1 = 2,$
$F_4 = F_3 + F_2 = 2 + 1 = 3,$
$F_5 = F_4 + F_3 = 3 + 2 = 5,$
$F_6 = F_5 + F_4 = 5 + 3 = 8,$
$\ldots$

Linear recurrences

Many counting problems yield recurrent equations.

While there's no general method to solve all recurrences, there are specific methods for common classes, for example linear recurrences.

General linear recurrence (of k-th order) has the form
$A_0 = m_0$, $A_1 = m_1$, $A_{k-1} = m_{k-1}$,
$A_{n+k} = b_1 A_{n+k-1} + b_2 A_{n+k-2} + ... + b_k A_n + f(n)$,
where m_i and b_j are constants and $f(n)$ is an arbitrary function on n.

Linear homogeneous recurrence of first order

Let's first consider the special case
$A_0 = m_0$,
$A_{n+1} = b_1 A_n$.

Here it is easy to see the general solution is
$A_n = m_0 b_1^n$.

Linear homogeneous recurrence of second order

$A_0 = m_0,\ A_1 = m_1,$
$A_{n+2} = b_1 A_{n+1} + b_2 A_n.$

By analogy with the previous case, we'll start from looking for solution of the form $A_n = q^n$. Substituting into the recurrence, we get

$$q^{n+2} = b_1 q^{n+1} + b_2 q^n$$

$$q^n (q^2 - b_1 q - b_2) = 0$$

Now the characteristic equation $q^2 - b_1 q - b_2 = 0$ may have
(a) two distinct solutions $q_1 \neq q_2$, or
(b) two coinciding solutions $q_1 = q_2$.

(a) In case of two distinct solutions $q_1 \neq q_2$, we can see that any linear combination

$$A_n = c_1 q_1^n + c_2 q_2^n$$

also satisfies the general recurrent equation, and we can use the boundary conditions to derive the equations

$$\begin{cases} c_1 + c_2 & = m_0 \\ c_1 q_1 + c_2 q_2 & = m_1 \end{cases}$$

for finding the c_1 and c_2 to get the particular solution that satisfies also the boundary conditions.

(b) In case of coinciding solutions $q_1 = q_2$, we can see that any linear combination

$$A_n = c_1 q_1^n + c_2 n q_1^n$$

also satisfies the general recurrent equation, and we can use the boundary conditions to derive the equations

$$\begin{cases} c_1 & = m_0 \\ c_1 q_1 + c_2 q_1 & = m_1 \end{cases}$$

for finding the c_1 and c_2 to get the particular solution that satisfies also the boundary conditions.

For Fibonacci numbers, the characteristic equation is
$q^2 - q - 1 = 0$,
from which we get
$q_1 = \frac{1+\sqrt{5}}{2}$, $q_2 = \frac{1-\sqrt{5}}{2}$.

From the boundary conditions we then have
$c_1 + c_2 = 0$, $c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} = 1$,
which yields
$c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$.

Therefore, the general expression for Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Linear homogeneous recurrences of higher order can be solved following the same pattern:

- first we extract the (k-th degree) characteristic equation from the recurrent rule and solve it;
- if the characteristic equation has solutions $q_1, \ldots, q_s$ with multiplicities $k_1, \ldots, k_s$ (where $k_1 + \ldots + k_s = k$), the general solution has the form

$$\begin{aligned} A_n = &(c_{1,0} + c_{1,1}n + \ldots + c_{1,k_1-1}n^{k_1-1})q_1^n + \\ &(c_{2,0} + c_{2,1}n + \ldots + c_{2,k_2-1}n^{k_2-1})q_2^n + \\ &\ldots + \\ &(c_{s,0} + c_{s,1}n + \ldots + c_{s,k_s-1}n^{k_s-1})q_s^n, \end{aligned}$$

  where the values of the multipliers $c_{i,j}$ can be found using the boundary conditions.

Suppose, for example, that the rabbits would breed as before, but would die after living for a year.

Then the recurrent relation would become
$G_{n+12} = G_{n+11} + G_{n+10} - G_n$
and the characteristic equation
$q^{12} - q^{11} - q^{10} + 1 = 0.$

To make the system well-defined we would also need to write out all G_n for $n = 0 \dots 11$ as boundary conditions.

The expressions for q_i and $c_{i,j}$ arising from this set of equations would be quite horrific, though.

Non-homogeneous recurrences differ by having an extra member in the recurrent relation:
$A_0 = m_0$, $A_1 = m_1$, $A_{k-1} = m_{k-1}$,
$A_{n+k} = b_1 A_{n+k-1} + b_2 A_{n+k-2} + ... + b_k A_n + f(n)$,

These can be solved using a three-step process:

- find the general solution A'_n of the corresponding homogeneous recurrence;
- find a particular solution A''_n of the non-homogeneus recurrence (any one will do);
- the general solution of the non-homogeneus recurrence is then $A_n = A'_n + A''_n$.

Suppose, for example, that instead of just buying one pair of rabbits on month 1, the farmer will buy n new pairs of rabbits on month n, for every n.

Then the recurrent relation would become
$H_0 = 0,\ H_1 = 1,$
$H_{n+2} = H_{n+1} + H_n + (n+2).$

We already know that the general solution of the corresponding homogeneous relation is
$H'_n = c_1 q_1^n + c_2 q_2^n$, where
$q_1 = \frac{1+\sqrt{5}}{2},\ q_2 = \frac{1-\sqrt{5}}{2}.$

We will now look for the particular solution of the non-homogeneous relation among expressions that follow the general form of the non-homogeneous member, that is $H''_n = \alpha n + \beta$.

Substituting into the recurrence, we get
$\alpha(n+2) + \beta = \alpha(n+1) + \beta + \alpha n + \beta + (n+2)$, or
$(\alpha+1)n + (\beta - \alpha + 2) = 0$.

This expression must hold for all n, therefore
$\alpha + 1 = 0$, or $\alpha = -1$, and
$\beta - \alpha + 2 = 0$, or $\beta = -3$, or
$H''_n = -n - 3$.

So, we have the general solution
$H_n = c_1 q_1^n + c_2 q_2^n - n - 3$, where
$q_1 = \frac{1+\sqrt{5}}{2}$, $q_2 = \frac{1-\sqrt{5}}{2}$

Finally, the boundary conditions
$H_0 = 0$, $H_1 = 1$ yield
$c_1 = \frac{7+3\sqrt{5}}{2\sqrt{5}}$, $c_2 = \frac{-7+3\sqrt{5}}{2\sqrt{5}}$,
and thus the complete solution is

$$H_n = \frac{7+3\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{-7+3\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n - n - 3.$$

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THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?